MATHEMATICAL ASPECTS of the QUANTUM THEORY OF FIELDS

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PREFACE

Five papers on the mathematical aspects of the quantum theory of fields previously printed in *Communications on Pure and Applied Mathematics* are republished in the present volume.

As stated in the introduction to these papers, they represent an attempt towards a mathematically consistent treatment of basic notions and simple problems in the quantum theory of fields. Specifically, we shall deal with problems in which the forces acting on the field depend linearly on the field quantity and in which, therefore, this quantity satisfies a linear equation; these linear problems are treated rather exhaustively. Many more or less straightforward conclusions, which in the physical literature are implied but not always explicitly stated, are here developed in great detail.

It is true that nonlinear interactions occupy at present the center of interest in the quantum theory of fields. A mathematically satisfactory treatment of such nonlinear problems could not, however, be given. It may be doubted whether nonlinear interaction in the form so far assumed admits such a treatment; in fact it has frequently been suggested that fundamentally different laws of nonlinear interaction should be adopted.

The Introduction contains a description of the content of the five parts. Since these five parts were written successively, the actual content of the later parts does not always exactly coincide with what has been announced in the introduction. The point of view has gradually changed, and some inconsistency has crept into the notations. The most important deviations are listed in "Corrections and Comments." In particular, Part VI has been omitted.

The appendix contains a Lorentz invariant treatment of the problem dealt with in Part V, on the basis of work by B. Zumino which will appear in forthcoming papers.

K. O. FRIEDRICHS

CONTENTS

Preface iii

Introduction 1

Remarks about Functional and Spectral Representation 4

Part I. **Field Operators** 11

1. Simultaneous Spectral Representation of Infinitely Many Operators . . . 11
2. Commutation Rules and Improper Operators 13
3. The Differential Equations 17
4. The Energy Integral 20
5. Motivation of the Configuration Space Representation 25

Part II. **Particle Representation** 28

6. Biquantization 28
7. Remark on the Occupation Number Representation 35
8. Annihilation and Creation Operators 37
9. Time Variation of Annihilation and Creation Operators and Representation of Field Operators 45
10. Trace Operators 46
11. Oscillators 51
12. Hermite Functionals and Integration over the Hilbert Space 52

Bibliography to Parts I and II 63

Part III. **Boson Field in Interaction with a Given Source Distribution** 65

13. Expectation Values of the Energy and the Number of Bosons 65
    - Representations 67
    - Operators and Differential Equations 68
    - Asymptotic Expectation Values 72
    - Modified Particles 77
14. Particle Representation of the States of the Boson Field Modified by a Source Distribution 79
    - First Form of the Operator T 83
    - Second Form of the Operator T 86
    - Modified Vacuum State 88
    - Transformation to the Modified Particle Representation 89
    - Probabilities in General 91
    - Derivation 92
    - Probabilities in Special Cases 93
15. Transition Probabilities 95
    - Transition Probabilities for the Vacuum State 97
    - General Transition Probabilities 99
    - Perturbation 102
    - Method of Spectral Transformation 103

- 16. Boson Fields under the Influence of a Source Distribution Which Varies in Time . . . 104
  - Infinitely Slow Switch-on . . . 107
  - Removing Sinks from Sources . . . 109
  - Switching Off . . . 109
  - Lorentz Invariant Formulation . . . 110
- 17. Modified Vacuum States . . . 113
  - Influence of a Source Distribution . . . 115
  - Probability Distribution of the Energy . . . 118

Bibliography to Part III . . . 120

Supplementary References to Parts I and II . . . 120

***Part IV.* Occupation Number Representation and Fields of Different Kinds** . . . 121

- 18. Occupation Number Representation . . . 121
  - Particle and Occupation Representation . . . 123
  - Occupation Functionals . . . 125
  - Correspondence between Occupation Functionals and Particle Representers . . . 128
  - Representation by Occupation Functionals . . . 129
  - Different Forms of the Occupation Representation . . . 130
  - Occupation Functions of a Discrete Variable . . . 130
  - Annihilation and Creation Operators . . . 132
  - Formal Operations . . . 134
  - Biquantized Operators . . . 136
  - Modified Vacuum State . . . 137
  - Expectation Values . . . 138
  - Equidistribution State . . . 139
- 19. Myriotic and Amyriotic Fields . . . 139
  - Functionals of the Second Type . . . 142
  - Annihilation and Creation Operators . . . 145
  - The Functionals $\bar{f}(\nu)$ . . . 147
  - Representation by Functionals $\phi(\nu)$ . . . 148
  - Proof that Myriotic Fields Possess No Vacuum States . . . 149
  - Equidistribution State . . . 151
  - Expectation Values . . . 152
  - Infrared Catastrophe . . . 153
  - Occupation Functions of a Discrete Variable . . . 153
  - Myriotic Field in a Box . . . 155
- 20. Probabilities and Expectation Values for the Equidistribution State . . . 156
  - Evaluation of $I_w(\tau)$ for Polynomials $\tau(\nu)$ . . . 157
  - Evaluation of the Expression $I_w(\tau)$ by Complex Integration . . . 160
  - Conditions on $\lambda(s)$, $w(\mathfrak{C})$, and $h(z)$ . . . 162
  - Evaluation of Probabilities . . . 164
  - Saddle Point Method . . . 166
  - Conditioned Equidistribution State . . . 166
- 21. Occupation Number Representation for Fermion Fields . . . 168
  - Representation of the First Type . . . 169
  - Representation of the Second Type . . . 173
  - Equidistribution State . . . 174
  - Definition of the Functional $F(\nu)$ . . . 175
  - Evaluations for the F-Equidistribution State . . . 176
  - Properties of the Projector F . . . 179
  - Partly Myriotic Fields . . . 181

Bibliography to Part IV . . . 183

Part V. **Fields Modified by Linear Homogeneous Forces** 185

22. Boson Fields under the Influence of Spring Forces 185
    Adjoint and Conjugate Operators 186
    Various Types of Problems 187
    Modified and Unmodified Particle Representation 188
    Modified Energy Operator for Single Particles 189
    Modified Creation and Annihilation Operators 190
    Remarks about Energy Operators 191
    Modified Particle Representation 192
    Canonical Transformations 193

23. General Homogeneous Linear Transformation of Creation and Annihilation Operators 194
    Pseudo-biquantized Operators 197
    First Commutator Identity 199
    Exponential Function of Pseudo-biquantized Operators 200
    First Similarity Rule, First Form of the Operator T 201

24. E-ordering of the Canonical Transformation 201
    Second Commutator Identity 202
    Composition Rule 203
    Second Form of the Canonical Transformation 204
    Modified Vacuum State 207
    First Decomposition 209
    Relations between E, F, G, and Y 210
    Trace Relations 211
    Conditions for the Existence of the Canonical Transformation 212

25. Third and Fourth Form of the Canonical Transformation 214
    Third and Fourth Form of the Transformation T 214
    Second Decomposition 215
    Final Form of the Transformation T' 217
    Identification of the Third and Fourth Form of T 217
    Second and Third Similarity Rule 219
    Composition Rule for Biquantized Operators 220
    Identity of the Fourth and the Second Form of the Operator T . . . 221

26. Application to Boson Fields 222
    Reduction of the Quantized to the Unquantized Field Problem 223
    Conditions for the Existence of the Canonical Transformation 224
    Special Cases 224

27. Transition Operator. Scattering Operator 227
    Method of Spectral Transformation 227
    Transition Operator 228
    Direct Method 229
    Properties of the Transition Operator 230
    Time Variation of the Vacuum State 231
    Asymptotic Transition Probabilities 232
    Scattering Operator 233
    Scattering Operator According to Yang and Feldman 234
    Asymptotic Field for Scattering Operator 235
    Justification 235
    Interpretation 237

28. A Modified Electron-Positron Field 239
Dirac Electron 240
Transformation of the Quantum Variables 241
Electron-Positron Field 245
A Modified Electron-Positron Field 247
Linear Transformation of Operators A into Operators B 247
Modified Vacuum State 249
Vacuum Transition Probability 249
Perturbation Approximation 251
Bibliography to Part V 251

Appendix: Lorentz Invariant Treatment of Boson Fields 257
29. Unquantized Field 257
Momentum Representation 258
Equations for Wave Amplitudes 261
Inverse Operators 262
Inner Products 263
30. Boson Field Subject to Homogeneous Forces 265

Comments and Corrections 269

Supplementary Bibliography 272

Mathematical Aspects of the Quantum Theory of Fields

Introduction

The present expository paper is based on lectures given in a Seminar on Mathematical Physics at New York University.[1] It is primarily addressed to mathematicians who are familiar with the fundamental concepts of quantum theory of single particles and who would like to learn which mathematical concepts are involved in the simplest problems of field quantum theory.

It is difficult for a mathematician to gather such information by reading papers and books addressed to physicists. It is not at all lack of rigor in the mathematical deductions which creates the difficulty; it is rather that the mathematical terms employed are not always defined precisely and that often their physical significance is not explicitly explained.

To be sure, only through the whole context of physical experience and theory could the notions of physics be properly defined. According to Niels Bohr, any attempt at a sharp definition of physical concepts must even violate their real physical meaning. Therefore, the mathematician's desire for a deductive presentation of physical theory cannot be satisfied in principle. On the other hand, it seems justified to strive for a precise definition of the intrinsic mathematical meaning of mathematical notions employed in field quantum theory.

The amount of physical insight gained in field quantum theory is great and many new mathematical ideas have been advanced by physicists in this connection. Naturally, only a part of this development can be discussed in this exposition. Our primary object is to present basic sections of field quantum theory in a consistent mathematical language without carrying out all mathematical deductions with complete rigor.

One of the important terms used in field quantum theory is Dirac's delta function; it enters nearly all basic formulas of field quantum theory. It is well understood that an equation involving the function $\delta(x)$ linearly is the "symbolic" expression of an equation involving only proper functions; how to give the delta function a precise meaning is thus rather obvious.

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[1] The author is greatly indebted to Dr. H. E. Moses for valuable assistance, in particular, for acquainting him with idiom and literature in field quantum theory.

It is characteristic of relations in field quantum theory that they connect the delta function with operators which depend on the point x in space. Therefore these operator functions also have only a "symbolic" meaning. Relations in which such symbolic operator functions occur linearly can be transformed into relations which involve only proper operators. It can be shown—which is not at all obvious—that there really exist proper operators which satisfy these relations. On the other hand, we find relations in field quantum theory in which the symbolic operator functions enter in a nonlinear way. It is far from obvious whether or not these relations can be transformed into relations involving only proper operators. Nevertheless, some of these relations can be given a meaning; certain others, however, cannot be saved.

Questions about the improper operators occurring in the field quantum theory of Heisenberg and Pauli (1929) will be raised in Part I and answered in Part II, with the aid of the "particle representation" of fields. This particle representation and its connection with the so-called "second quantization" was clearly formulated by Fock in 1932, and has always been used by physicists, though, perhaps, not too consistently.

The lack of full exploitation of the possibilities inherent in the particle representation seems to be connected with the fact that many physicists prefer to treat fields enclosed in a finite "box", rather than those extending over the whole space. Since the energy of such an enclosed field has a pure point spectrum it is possible to describe the field as an assembly of "oscillators" and to characterize its states in terms of "occupation numbers". Expressions derived on this basis depend on the size of the box and may possess limits as the size of the box increases indefinitely. It seems more appropriate to deal immediately with the field which extends over the infinite space and thus to derive the limit expressions directly. Such a direct approach is actually far simpler than the detour via the box, while the description in terms of oscillators and occupation numbers then becomes somewhat unwieldy.

We can avoid oscillators in describing the states of fields by introducing Hermite polynomials of a kind different from the standard Hermite polynomials used when fields are described in terms of oscillators. These modified Hermite polynomials are functionals of elements in Hilbert space. By expansion in terms of these polynomials we can characterize a class of functionals which are quadratically integrable over this Hilbert space and can be used to represent the states of the field. By characterizing this class of functionals we shall, incidentally, give an answer to a question frequently asked, namely, how to integrate over the Hilbert space.

In Part III we shall take up one of the simplest problems in field quantum theory, namely the effect of a given distribution of charges on a field. This problem can be solved explicitly by methods which are somewhat more concise than those available in the literature. Among other things, we shall determine explicitly the probability that a field which originally consisted of no particles will consist of n particles after an infinitely long time. We shall also see that no scattering operator in the strict sense can be associated with this problem.

In Part IV we shall show that a description of the field in terms of "occupation numbers", though somewhat unwieldy, can nevertheless be introduced, even if the spectrum of the energy is not discrete. It will be seen that there are two types of such occupation number representations; only one of them is equivalent with a particle representation. Accordingly, there are different—non-equivalent—realizations of the basic field operators, and consequently different—non-equivalent—kinds of fields, a fact which seems worth noticing. The realization that occurs in nature appears to be the one that admits the description of the state of an undisturbed field in terms of particles. Nevertheless, the second type of occupation number representation has also its place; it is to be employed if the field is modified by a distributed charge of a total amount different from zero.

In Part V we shall deal with the effect of distributed charges which depend linearly on the field potential or, more generally, with problems in which the disturbed differential equations remain linear and homogeneous in the field potential. First we shall treat fields which consist of assemblies of particles with symmetric wave functions. The problem can be solved if appropriate smoothness conditions are imposed on the disturbance. The main objective is to determine the probability that a certain number of these particles will be found at a certain time t when this number is known at an initial time. We shall show that this transition probability—and its asymptotic value as $t \rightarrow \infty$—can be calculated explicitly.

Greater difficulties arise in dealing with fields which consist of assemblies of particles with antisymmetric wave functions. We shall treat the case in which the undisturbed field is a matter field composed of Dirac electrons. We shall discuss the representation of the states of such fields in terms of electrons and positrons, each having a positive energy. Certain infinite terms, sometimes omitted with special excuses, do not occur when this representation is used. Furthermore we shall show that the transition probabilities can be derived from the solution of an integral equation which can be attacked by manageable approximation procedures.

In Part VI we shall deal with simple problems similar to those which concern the interaction of matter with light. We shall indicate how to determine the diagonal representation of the energy operator directly without first solving the Schrödinger equation for a time-dependent state. We shall employ a perturbation procedure in two steps. The desired scattering operator will—in second order—be found in the second step which involves no change in the number of particles; the first step, which does involve such a change, cannot—in second order—be associated with a scattering operator. Certain terms occurring in this procedure will turn out to be infinite. In particular we shall discuss the infinite term which has been removed by "mass renormalization."

The most important guide in setting up the correct basic equations of field quantum theory has been the requirement that they be invariant under Lorentz transformation. Field quantum theory cannot properly be understood unless these matters have been thoroughly discussed. Nevertheless, we shall merely touch upon these questions. For, once the Lorentz invariance of the basic equations is assured, it need no longer be exhibited at every step in the solution of specific problems.

Our main mathematical tool will be the representation of states by functions, and of operators by functional operators. We shall give a short discussion of these notions in the following section. For reasons explained there we shall not use Dirac's elegant notation.

Remarks about Functional and Spectral Representation

Spectral—or diagonal—representation is certainly implied in Dirac's notation in a formal way. The basic mathematical facts concerning this representation were already contained in the theory of Hellinger, (1907) [18] in conjunction with Hilbert's theory of bounded operators (1906) [17]. Nevertheless, our main device, to interpret spectral representation as functional representations, is not explicitly formulated in the work on operators in Hilbert space written for mathematicians. It seems therefore advisable to sketch this matter briefly.[2]

We consider a linear manifold of "elements" Φ (which correspond to states or state vectors) forming a Hilbert space $\mathfrak{H}$ with the inner product $(\Phi, \Phi^{(1)})$, of two elements $\Phi, \Phi^{(1)}$, and the norm $\| \Phi \| = (\Phi, \Phi)^{1/2}$.

A system of orthogonal "coordinate elements" Φ_σ, $\sigma = 1, 2, \cdots$ is said to be complete if every element Φ in $\mathfrak{H}$ admits an expansion of the form

$$\Phi = \sum\nolimits_\sigma c_\sigma \Phi_\sigma ,$$

in which

$$c_\sigma = (\Phi_\sigma , \Phi).$$

The element Φ is completely characterized by the system of numbers c_σ. We may just as well consider c_σ as a function of the variable σ with the domain $\sigma = 1, 2, \cdots$, and say that the element Φ is represented by the function c_σ as its "representer."[3] We express this correspondence symbolically by

---

[2] A short derivation of the spectral representation from the spectral resolution will be given in a subsequent paper.

[3] See *Webster's New International Dictionary of the English Language*, Second Edition, p. 2114.

$$\Phi \leftrightarrow \{c_\sigma\}.$$

Because of the orthogonality of the elements Φ_σ the inner product of two elements Φ and $\Phi^{(1)}$ is given by a series

$$(\Phi, \Phi^{(1)}) = \sum_\sigma \bar{c}_\sigma c_\sigma^{(1)} \rho_\sigma ;$$

the positive number

$$\rho_\sigma = (\Phi_\sigma , \Phi_\sigma)$$

could be made equal to one by norming the elements Φ_σ appropriately. Every "function" c_σ for which $\sum_\sigma \rho_\sigma \mid c_\sigma \mid^2 < \infty$ represents an element Φ in $\mathfrak{H}$.

Suppose, in particular, that the elements Φ_σ are eigenelements of an operator Λ, so that there are "eigenvalues" λ_σ such that

$$\Lambda\Phi_\sigma = \lambda_\sigma\Phi_\sigma .$$

Then, evidently the validity of the relation

$$\Lambda\Phi = \sum_\sigma c_\sigma\lambda_\sigma\Phi_\sigma$$

can be expressed by the formula

$$\Lambda\Phi \leftrightarrow \{\lambda_\sigma c_\sigma\}.$$

In other words, application of the operator Λ is represented through multiplication by the function λ_σ of σ. The representation $\Phi \leftrightarrow \{c_\sigma\}$ is then said to furnish a "diagonal" or "spectral" representation of the operator Λ.

It is frequently desirable to express the elements Φ in terms of coefficients c which depend on a continuous variable s instead of a discrete variable σ. One then is tempted to write down an expansion of the form

$$\Phi = \int c(s)\Phi_s \, ds.$$

As is well known, this cannot be done in a proper sense; for, there are no elements Φ_s in the Hilbert space which can be used as coordinate elements such that an expansion of the indicated type exists. To facilitate operations one might, of course, introduce "improper elements" Φ_s as convenient symbols. In all considerations, however, in which clarification of the proper mathematical meaning of the terms involved is the objective, the issue would be hidden if such "symbolic elements" were used as basic notions from the start.

To obtain the desired results, it is perfectly sufficient to indicate that the elements Φ are *represented* by functions $c(s)$. By this we mean that to each element Φ of $\mathfrak{H}$ a function $c(s)$ of a certain class, its "representer," is assigned in a one-to-one way such that to the linear combination $a_1\Phi^{(1)} + a_2\Phi^{(2)}$ of two elements $\Phi^{(1)}$, $\Phi^{(2)}$ with the representers $c_1(s)$, $c_2(s)$ the function $a_1c_1(s) + a_2c_2(s)$ is assigned. We indicate this representation by the formula

$$\Phi \leftrightarrow \{c(s)\}.$$

Such a functional representation seems to be the most appropriate tool for working with specific operators in a Hilbert space. In fact, whenever specific operators in a Hilbert space have been described, the elements of the Hilbert space were given as functions. The notion of function is here to be taken in a sufficiently wide sense, so that a sequence, for example, is considered a function of the subscript.

The argument s of the representer $c(s)$ may stand for a single real variable, a finite set of real variables, $s = \{s_1, s_2, \cdots\}$, or even infinitely many real variables. The number of variables, if infinite, need not be denumerable; in fact, we shall discuss a particular type of representation occurring in field quantum theory in which the number of variables is not denumerable.

Not every representation of states by functions will be admitted; an important condition will be imposed. In case the variable s is restricted to integer values σ, this condition implies the orthogonality of the coordinate states Φ_σ. In order to formulate this condition in general, the notion of *measure* must be introduced.

Let s be a single variable; we may introduce a measure function $m(s)$ as any non-decreasing function of s. This function may jump at denumerably many points at most; for each jump point an upper and lower value $m^{\pm}(s)$ is defined. To every interval Δs the measure $\Delta m(s) = m^{\pm}(s^{(2)}) - m^{\pm}(s^{(1)})$ is assigned; here $s^{(1)}$ and $s^{(2)}$ are the endpoints of the interval and the value of $\Delta m(s)$ depends on whether or not an endpoint is included in the interval. An open interval Δs for which $\Delta m(s) = 0$ is said to be outside of the "basic domain" $\mathfrak{D}$ of the representers. In other words, the basic domain is defined as the closed complementary set of the set covered by open intervals Δs for which $\Delta m(s) = 0$.

The notion of measure function can easily be carried over to the case that s represents a finite number of variables, and under certain circumstances also to the case that s represents infinitely many variables. In this case the intervals are replaced by cells, such as the open cells

$$s_1^{(1)} < s < s_1^{(2)}, \qquad s_2^{(1)} < s < s_2^{(2)}, \quad \cdots .$$

The basic domain $\mathfrak{D}$ of the representers is defined as the closed complement of all open cells Δs for which $\Delta m(s) = 0$.

With the aid of the measure function the integral

$$\int_{\mathfrak{D}} |\, c(s) \,|^2 \, dm(s)$$

can be defined for an appropriate class of functions $c(s)$. The manifold of functions $c(s)$ for which this integral is finite, forms a linear space $\mathfrak{K}$, provided a function $c(s)$ is said to be "equal" to zero if this integral vanishes; and two functions $c(s)$ are said to be "equal" to each other if their difference "equals" zero. Evidently, two functions $c(s)$ which differ only outside of the basic domain, or more generally in a set of "measure" zero, are "equal" to each other. How a function $c(s)$ is defined in such a set is not relevant.

We now require that the functions $c(s)$ which represent the elements Φ of the Hilbert space $\mathfrak{H}$ are just the functions of such a class $\mathfrak{K}$, provided that we stipulate that functions $c(s)$ which are "equal" to each other are counted as one function in the one-to-one assignment of functions to states.

The condition mentioned above, to be imposed on the representation, is that the relation

$$(\Phi, \Phi^{(1)}) = \int_{\mathfrak{D}} \bar{c}(s)c^{(1)}(s)\, dm(s)$$

should hold for two elements Φ, $\Phi^{(1)}$ in $\mathfrak{H}$ with the representers $c(s)$ and $c^{(1)}(s)$.

It follows from this formula that elements with representers vanishing outside of any disjoint cells are orthogonal to each other. If a cell with positive measure consists of just one point s_0 the function $c_0(s)$ which equals 0 for $s \neq s_0$, and which equals 1 for $s = s_0$, represents an element Φ_0 which may be called a "coordinate element". Any two coordinate elements associated with different values s_0 are therefore orthogonal.

One of the main problems of the theory is to find a functional representation of the elements of $\mathfrak{H}$ such that a given operator acting on these elements can be described in a simple manner in terms of the representers.

Let Λ be an operator acting on the elements Φ of $\mathfrak{H}$ or of a subspace $\mathfrak{D}(\Lambda)$ of $\mathfrak{H}$, called the "domain" of Λ. Then the representer $c'(s)$ of the state $\Lambda\Phi$ will be obtained from the representer $c(s)$ of the element Φ by applying a certain functional operator, its "representer", which we denote by Λ^S. I.e.

$$c'(s) = \Lambda^S c(s),$$

or

$$\Lambda\Phi \leftrightarrow \{\Lambda^S c(s)\}.$$

Suppose there exists a function $\lambda(s)$ such that the functional operator Λ^S consists merely in multiplying by $\lambda(s)$

$$\Lambda^S = \lambda(s)$$

or

$$\Lambda\Phi \leftrightarrow \{\lambda(s)c(s)\}.$$

We then say the representation of the elements Φ by the functions $c(s)$ furnishes a *diagonal* or *spectral representation* of the operator Λ. One of the primary aims in dealing with operators in Hilbert space, is to find such spectral representations.

The representers $c(s)$ of the particular representation which "diagonalizes" the operator Λ will be called the "spectral representers" for Λ; the associated measure $m(s)$ will be called the "spectral" measure. Certainly,

$$\int \lambda^2(s) \mid c(s) \mid^2 dm(s) < \infty$$

for the spectral representers of the elements Φ of the domain $\mathfrak{D}(\Lambda)$ of Λ, i.e. of the elements Φ for which Λ is defined. This domain will be called "maximal" and the operator Λ will be called "spectrally representable" in $\mathfrak{D}(\Lambda)$ provided that every element for whose representer $c(s)$ the integral $\int \lambda^2(s) \mid c(s) \mid^2 dm(s)$ is finite belongs to $\mathfrak{D}(\Lambda)$.

To illustrate the notion of spectral representation we consider the case in which the spectral measure function $m(s)$ has a "jump" at the value s_0, i.e. we assume that the measure $\Delta m(s)$ of the set Δs which consists only of the point s_0 is not zero. The value $\lambda_0 = \lambda(s_0)$ is said to be a "point eigenvalue" of Λ. The function $c_0(s)$ which equals zero for $s \neq s_0$ but is different from zero for $s = s_0$ is then quadratically integrable, and thus belongs to the class $\mathfrak{K}$. Evidently for the element Φ_0 which is represented by this function $c_0(s)$ the relation

$$\Lambda\Phi_0 - \lambda_0\Phi_0 \leftrightarrow \{\lambda(s)c_0(s) - \lambda_0 c_0(s)\} = 0$$

holds, whence

$$\Lambda\Phi_0 = \lambda_0\Phi_0 .$$

In other words, the state Φ_0 is an eigenstate of the operator Λ with the eigenvalue λ_0. The notions of eigenvalue and eigenelement in the proper sense can therefore be derived in a simple way from the general notion of spectral representation. On the other hand we see that the notion of an eigenelement associated with an eigenvalue λ in the spectrum which is not a point eigenvalue has no proper meaning.

The basic domain $\mathfrak{D}$ of the spectral representers $c(s)$ for an operator Λ is called the "spectrum" of the operator Λ. In other words, the "exterior" of the spectrum is defined as the set covered by all open intervals Δs of "spectral" measure zero, then the spectrum itself is defined as the complement of this exterior. This spectrum may partly or completely consist of point eigenvalues. The closure of what remains of the spectrum after removal of the point eigenvalues is called the continuous spectrum. Little simplification would be gained by treating the discrete and continuous parts of the spectrum separately. All types of spectra are easily treated in a unified way if one adopts the general notion of measure function.

Many advantages can be gained from spectral representations. It enables one, for example, to define the function of an operator in a very simple way. The function $f(\Lambda)$ of the operator Λ is simply defined as the operator which transforms the element Φ with the spectral representer $c(s)$ into the element with the spectral representer $f(\lambda(s))c(s)$,

$$f(\Lambda)\Phi \leftrightarrow \{f(\lambda(s))c(s)\}.$$

It should be mentioned that the spectral representation of an operator is not uniquely determined; for, instead of the variables $s = \{s_1, s_2, \cdots\}$, any functions of them may be introduced. It is sometimes possible to do this in such a way that one of the variables, e.g. s_1, is the eigenvalue λ itself, $\lambda(s) = s_1$; then the variables $s_2, \cdots$ may be called "accessory" variables.

A representation $\Phi \leftrightarrow c(s)$ is said to be a simultaneous spectral representation of a set of commuting operators $\Lambda_1, \Lambda_2, \cdots$ if there are functions $\lambda_1(s), \lambda_2(s), \cdots$ such that

$$\Lambda_1\Phi \leftrightarrow \{\lambda_1(s)c(s)\}, \qquad \Lambda_2\Phi \leftrightarrow \{\lambda_2(s)c(s)\}, \cdots .$$

If the $\lambda_1, \lambda_2, \cdots$ can be chosen as variables so that $\lambda_1 = s_1$, $\lambda_2 = s_2, \cdots$ while no accessory variables occur, the set of commuting operators $\Lambda_1, \Lambda_2, \cdots$ will be called "complete". In this case the system of operators will simply be denoted by S so that we may write

$$S\Phi \leftrightarrow \{sc(s)\}.$$

It can be shown that an operator which commutes with all operators of a complete set is a function of these operators.

The conditions under which an operator Λ possesses a spectral representation are well known. Suppose that the operator is bounded, i.e. suppose that a number C exists such that

$$\| \Lambda\Phi \| \leq C \| \Phi \|$$

for all elements Φ in $\mathfrak{H}$. In order that Λ be spectrally representable, it is then necessary and sufficient that Λ be symmetric in the Hermitian sense, i.e. that

$$(\Lambda\Phi, \Phi^{(1)}) = (\Phi, \Lambda\Phi^{(1)})$$

for all Φ, $\Phi^{(1)}$ in $\mathfrak{H}$, as follows from the theories of Hilbert and Hellinger. If, however, the operator Λ is not bounded, it cannot be defined in the whole space $\mathfrak{H}$, but only in an appropriate subspace $\mathfrak{D}(\Lambda)$, the "domain" of Λ. A significant necessary and sufficient condition that such an operator be spectrally representable in its domain was discovered by v. Neumann, E. Schmidt, and Stone. First, it is required that the operator Λ be Hermitian in its domain; i.e. the above symmetry relation should hold for all Φ, $\Phi^{(1)}$ in $\mathfrak{D}(\Lambda)$. A second requirement is imposed in order to make sure that the domain is not chosen too narrow; This domain should contain every element Φ to which there is an element Φ' such that the relation

$$(\Phi', \Phi^{(1)}) = (\Phi, \Lambda\Phi^{(1)})$$

holds for arbitrary elements $\Phi^{(1)}$ in $\mathfrak{D}(\Lambda)$. Evidently, $\Lambda\Phi = \Phi'$ for such an element Φ. An operator Λ satisfying these two conditions was called "hypermaximal" by v. Neumann, and "self-adjoint" by Stone. In order to avoid confusion with the term "self-adjointness" for differential operators, we prefer to use the term "hypermaximal". However, it will not be necessary to refer to this condition of hypermaximality in this paper; it is perfectly sufficient to assume that the operators are spectrally representable in their domain.

Operators are frequently called transformations in literature. We shall use these two terms for two different notions. We speak of an operator as a mapping of an element into another element; we speak of a transformation as a mapping of one functional representation into another.

The functional representer Λ^S of the operator Λ introduced above can symbolically be written as an integral operator

$$\Lambda^S c(s) = \int \Lambda(s, s')c(s')\, dm(s')$$

with a "kernel" $\Lambda(s, s')$, which in general is not a proper function of s and s', but has only a symbolic meaning. For example, if the operator Λ is a function $\lambda(S)$ of S the kernel of the integral operator Λ^S is $\Lambda(s',s'') = \lambda(s')\delta(s' - s'')$. In a similar way, transformations can be written symbolically as integral transformations. Dirac [2] prefers to work with these kernels because of various formal advantages.

One of the significant features of Dirac's notation [2] is that the functional representers of an element Φ are written as inner products of Φ and an appropriate other element. Dirac uses symbols such as $|a\rangle$ to denote an element—or state vector; the letter a here stands for a label chosen at convenience. An eigenelement—proper or improper—belonging to the eigenvalue s of a complete system of commuting variables S is then denoted by $|s\rangle$. The inner product of the element $|a\rangle$ with the eigenelement $|s\rangle$ is written as $\langle s|a\rangle$. This product may be considered as a function $c(s)$ of s; then $c(s) = \langle s|a\rangle$ is the S-representer of the element $|a\rangle$.

The expansion of the element $|a\rangle$ with respect to the eigenelements $|s\rangle$ of S is written as

$$|a\rangle = \int |s'\rangle\langle s'|a\rangle\, ds' + \sum_{s'} |s'\rangle\langle s'|a\rangle.$$

The integral in these formulas extends over the continuous spectrum of S and the series over its point spectrum. Of course, the two terms could be combined into one by using a measure function. The element $\Lambda|a\rangle$ produced from the element $|a\rangle$ by an operator Λ then possesses the expansion

$$\Lambda|a\rangle = \int \Lambda|s'\rangle\langle s'|a\rangle\, ds' + \sum_{s'} \Lambda|s'\rangle\langle s'|a\rangle,$$

whence

$$\langle s|\Lambda|a\rangle = \int \langle s|\Lambda|s'\rangle\langle s'|a\rangle\, ds' + \sum_{s'} \langle s|\Lambda|s'\rangle\langle s'|a\rangle.$$

The left member here is nothing but the representer $\Lambda^S c(s)$ of the element $\Lambda|a\rangle$; thus it is clear that $\langle s|\Lambda|s'\rangle$ is just the kernel $\Lambda(s,s')$ of the integral operator Λ^S.

We found Dirac's notation not suitable for our purpose. We felt it desirable to use a short notation for the functional representer of the operator Λ instead of writing it as an integral operator. Moreover, we did not want to use improper elements and improper kernels in spite of their formal advantages. We are not primarily concerned with formal relations but rather with investigating the legitimacy of the operations of field quantum theory. Although there is no doubt that the use of improper elements and kernels could be made legitimate, we felt that we could achieve our aim better without using them.

Part I. Field Operators

1. *Simultaneous Spectral Representation of Infinitely Many Operators*

The basic concepts of the field quantum theory, with which this exposition is concerned, were essentially developed by Heisenberg and Pauli in 1929 [11], following earlier work in this direction by Dirac, Jordan, Wigner and others (for a bibliography of earlier works see [1]). One of the features in which this theory differs in a mathematical respect from the quantum theory of single particles is the occurrence of an infinite set of operators which commute with each other. While the mathematical problem of the simultaneous spectral representation of a finite number of commuting operators can be handled almost as easily as that of a single operator, the problem of the simultaneous spectral representation of infinitely many commuting operators requires new mathematical considerations. We shall at first discuss this aspect of the theory.

Let us denote by Ξ any field quantity such as an electric-magnetic scalar or vector potential or a meson field quantity. According to the principles of quantum theory a Hermitian spectrally representable operator is assigned to such a quantity which will also be denoted by Ξ. (A spectrally representable operator is one that possesses a spectral representation, as explained in the introduction.) This operator operates on "states" or "state vectors" Φ. The manifold of all states forms a Hilbert space $\mathfrak{H}$. It is, however, not assumed that Ξ is defined in the full space $\mathfrak{H}$; it is sufficient that Ξ be defined in an appropriate dense subspace of $\mathfrak{H}$ as a spectrally representable operator. The possible discrete or continuous eigenvalues of the operator Ξ will be denoted by ξ.

A specific feature of the field quantum theory is that such an operator Ξ is assigned to every point $x = \{x_1, x_2, x_3\}$ of the space, and that these operators $\Xi = \Xi(x)$ are assumed to commute with each other.

The simultaneous spectral representation of a finite number of commuting operators $\Xi_1, \cdots, \Xi_n$ requires that the states Φ be represented by functions $\phi(\xi_1, \cdots, \xi_n)$ of the spectral variables $\xi_1, \cdots, \xi_n$, and, if necessary, of additional "accessory" variables, in such a way that the state $\Xi_\nu\Phi$ is represented by the function $\xi_\nu\phi(\xi_1, \cdots, \xi_n)$. In the following we assume that no accessory variables are needed. The system of operators $\Xi_1, \cdots, \Xi_n$ is then said to be complete. The numbers $\xi_1, \cdots, \xi_n$ are also called the eigenvalues of the operators $\Xi_1, \cdots, \Xi_n$. The system of these numbers may simply be denoted by

$$\xi = \{\xi_1, \cdots, \xi_n\}, \tag{1.1$_n$}$$

so that we may indicate the representations through

$$\Phi \leftrightarrow \phi(\xi), \tag{1.2$_n$}$$

$$\Xi_\nu\Phi \leftrightarrow \xi_\nu\phi(\xi). \tag{1.3$_n$}$$

The significance of the function $\phi(\xi)$ is that it is a probability amplitude; specifically, $|\phi(\xi)|^2$ is the density of the probability of finding that the quantity Ξ has the value ξ.

It is now clear how a simultaneous spectral representation of the infinite set of commuting operators $\Xi(x)$ should be formulated. The state Φ should be represented by a function ϕ of the variables ξ assigned to the points x, thus of infinitely many variables. (The infinitely many values of x here correspond to the n values $1, \cdots, n$ of the variable ν above). We can just as well say that ϕ should depend on the function $\xi = \xi(x)$. In other words, $\phi(\xi)$ should be a "functional". The representation

$$\Phi \leftrightarrow \phi(\xi) \tag{1.2}$$

of the state Φ by a functional $\phi(\xi)$ should then be such that the state $\Xi(x)\Phi$ is represented by the functional $\xi(x)\phi(\xi)$,

$$\Xi(x)\Phi \leftrightarrow \xi(x)\phi(\xi). \tag{1.3}$$

Here x is an arbitrarily chosen point. Since the value $\xi(x)$ of a function ξ at a point x is a functional of the function ξ, the product $\xi(x)\phi(\xi)$ also is such a functional.

In order to complete the spectral representation (1.2) of a set of a finite number, n, of commuting operators it is necessary to express the inner product

$(\Phi^{(1)}, \Phi^{(2)})$ of two states in terms of the representers $\phi^{(1)}(\xi)$, $\phi^{(2)}(\xi)$. This expression is an integral

$$(1.4)_n \qquad (\Phi^{(1)}, \Phi^{(2)}) = \int \overline{\phi^{(1)}}(\xi)\phi^{(2)}(\xi)\, dm(\xi)$$

over the n-dimensional ξ-space with the aid of a measure given by an appropriate function $m(S)$ of sets S in the ξ-space. The integral

$$\int_S |\,\phi(\xi)\,|^2\, dm(\xi)$$

over a set S then represents the probability of finding that the quantity Ξ has a value in S.

In order to complete the spectral representation (1.3) of a set of infinitely many operators $\Xi(x)$ analogously, we must also express the inner product $(\Phi^{(1)}, \Phi^{(2)})$ in terms of the representing functionals $\phi(\xi)$. In some sense or other, this expression would be an integral over the infinitely many dimensional space of the functions $\xi(x)$. If such an integral could be extended over a subset S of the space of all functions $\xi(x)$, one could determine the probability that a measurement of the quantity $\xi(x)$ would yield a value in the set S. How one can define such an integral, at least in the simplest and most important case will be discussed later on in Section 12.

2. *Commutation Rules and Improper Operators*

The specific operators $\Xi(x)$ which occur in the field quantum theory are, in general, characterized by the condition that they satisfy certain formal rules. Specifically, the following formal rule is stipulated for a large class of fields: There should be another field quantity, $\Pi = \Pi(x)$ (later identified with $d\Xi/dt$) such that the corresponding spectrally representable operators, also denoted by $\Pi(x)$, commute with each other while between Π and Ξ the identity

$$(2.1) \qquad \Pi(x')\Xi(x'') - \Xi(x'')\Pi(x') = -i\delta(x' - x'')$$

holds. Here $\delta(x)$ is Dirac's delta function. At the same time $\Xi(x')$ and $\Xi(x'')$ commute, as mentioned previously. Thus

$$(2.1)_1 \qquad \Xi(x')\Xi(x'') - \Xi(x'')\Xi(x') = 0,$$

$$(2.1)_2 \qquad \Pi(x')\Pi(x'') - \Pi(x'')\Pi(x') = 0.$$

By adopting commutation rule (2.1) we have restricted ourselves to a certain class of fields to which the electromagnetic and the meson fields belong. For other fields, such as the matter field, a commutation rule holds which differs from (2.1) in that a plus sign takes the place of the minus sign in its left member

Suppose the states are represented by functionals $\phi(\xi)$ which possess a functional derivative $\delta\phi/\delta\xi(x)$, defined by the condition, that for an arbitrary function $\gamma(x)$,

$$\int \frac{\delta\phi}{\delta\xi(x)}\gamma(x)\,dx = \lim_{\epsilon\to 0}\epsilon^{-1}[\phi(\xi+\epsilon\gamma)-\phi(\xi)]. \tag{2.2}$$

Then the derivative of the functional $\xi(x')\phi(\xi)$ is given by

$$\int \frac{\delta}{\delta\xi(x)}\xi(x')\phi(\xi)\gamma(x)\,dx = \int \xi(x')\frac{\delta}{\delta\xi(x)}\phi(\xi)\gamma(x)\,dx + \gamma(x')\phi(\xi),$$

so that

$$\frac{\delta}{\delta\xi(x)}\xi(x')\phi(\xi) = \xi(x')\frac{\delta}{\delta\xi(x)}\phi(\xi) + \delta(x'-x)\phi(\xi),$$

or, simply,

$$\frac{\delta}{\delta\xi(x)}\xi(x') - \xi(x')\frac{\delta}{\delta\xi(x)} = \delta(x'-x). \tag{2.3}$$

It is then seen that the operator $\Pi(x)$ defined by the representation

$$\Pi(x)\Phi \leftrightarrow -i\frac{\delta}{\delta\xi(x)}\phi(\xi) \tag{2.4}$$

together with the operator $\Xi(x)$ defined by (1.3) satisfies the commutation rule (2.1). Representation (2.4), given by Heisenberg and Pauli [11], is the natural analogue of Schrödinger's representation of the momentum of a single particle.

It is one of the aims of the present investigation to specify the class of functionals $\phi(\xi)$ which represent states Φ and to give a precise meaning to the representations (1.3) and (2.4). It will be seen that the functionals $\xi(x')\phi(\xi)$ and $\delta\phi(\xi)/\delta\xi(x)$ are not functionals of the same class as $\phi(\xi)$, so that the operators $\Xi(x)$ and $\Pi(x)$ are not in a proper sense operators acting on elements in a Hilbert space.

The improper character of the operators $\Xi(x)$ and $\Pi(x)$ can already be inferred from the commutation relation (2.1), since the symbolic function $\delta(x'-x'')$ occurs in its right member. Accordingly, it is clear that the operators $\Xi(x)$ and $\Pi(x)$ also have only a "symbolic" meaning. This symbolic meaning can be described. Guided by the notions advanced by L. Schwartz [24], one introduces two operators which depend linearly on functions $\gamma(x)$ of a class to be specified. That is to say, one introduces two linear functionals, $\Pi\{\gamma\}$ and $\Xi\{\gamma\}$, or, as Schwartz says, "distributions," which to each function γ assign spectrally representable operators. Of these operators one then requires that for any pair $\gamma'(x)$ and $\gamma''(x)$ of functions $\gamma(x)$ the identity

$$\Pi\{\gamma'\}\Xi\{\gamma''\} - \Xi\{\gamma''\}\Pi\{\gamma'\} = -i\int \gamma'(x)\gamma''(x)\,dx \tag{2.5}$$

holds. We maintain that formula (2.5) is the proper expression of the relation which is symbolically expressed through (2.1).

In order to justify this statement we write the linear functionals $\Pi\{\gamma\}$ and $\Xi\{\gamma\}$ in the form of integrals

$$\Pi\{\gamma\} = \int \Pi(x)\gamma(x)\,dx, \qquad \Xi\{\gamma\} = \int \Xi(x)\gamma(x)\,dx \tag{2.6}$$

with the aid of "symbolic" operators $\Pi(x)$, $\Xi(x)$, which have no meaning in themselves. Relation (2.5) then assumes the form

$$\begin{aligned} &\int \Pi(x')\gamma'(x')\,dx' \int \Xi(x'')\gamma''(x'')\,dx'' \\ &\quad - \int \Xi(x'')\gamma''(x'')\,dx'' \int \Pi(x')\gamma'(x')\,dx' = -i\int \gamma'(x)\gamma''(x)\,dx. \end{aligned} \tag{2.7}$$

In order to eliminate the arbitrary functions γ' and γ'' occurring in this identity, one may write its right hand side in the form

$$-i\int \gamma'(x)\gamma''(x)\,dx = -i\iint \gamma'(x')\delta(x' - x'')\gamma''(x'')\,dx'\,dx'' \tag{2.8}$$

and then strip off the symbols $\cdots \iint \cdots \gamma'(x')\gamma''(x'')\,dx'\,dx''$ from both sides. The result is formula (2.1). Thus it is seen that the commutation rule (2.1) is nothing but the symbolic expression of the identity (2.5).

The statement that the field operators $\Pi(x)$ and $\Xi(x)$ at a point x have no meaning in a proper sense admits of a physical interpretation: it is impossible to measure the field quantities Ξ and Π at a fixed point x. Instead, it is possible to measure the quantities $\int \Pi(x)\gamma(x)\,dx$, $\int \Xi(x)\gamma(x)\,dx$ involving an arbitrary function $\gamma(x)$ (of a certain class). One may consider these quantities as averages of the field quantities with the weight factor $\gamma(x)$. That field quantities of a fixed point cannot be measured while averages of them can, was stated by Bohr and Rosenfeld [16].

The operators $\Pi(x)$ and $\Xi(x)$ are symbolic operators in much the same sense as the Dirac δ-function is a symbolic function. In view of this situation one should not be surprised when the numerical evaluation of expressions such as the expectation values

$$(\Phi, \Xi(x)\Phi), \qquad (\Phi, \Xi^2(x)\Phi)$$

of the field quantities $\Xi(x)$, $\Xi^2(x)$ in certain states yields the value infinity. Some of the infinities that haunt field quantum theory may be interpreted as being of this nature. On the other hand, expressions such as the expectation value

$$\left(\Phi, \int \Xi(x)\gamma(x)\,dx\,\Phi\right)$$

of the proper operator $\int \Xi(x)\gamma(x)\,dx$, will turn out to be finite, at least for a dense subspace of states Φ. Consequently, since the expression

$$\left(\Phi, \int \Xi(x)\gamma(x)\,dx\,\Phi\right)$$

can be written as

$$\int \gamma(x)(\Phi, \Xi(x)\Phi)\,dx,$$

the expectation value $(\Phi, \Xi(x)\Phi)$, considered as a function of x will be integrable and hence be finite at least almost everywhere; thus in a certain sense this expectation value can be given a meaning.

We can, however, give no meaning to the expectation value of the operator $\Xi^2(x)$. The square of a merely symbolic entity is meaningless on the face of it (just as $\delta^2(x)$ is undefined) and only for very special symbolic functions can such a square be defined. In fact, no meaning can be given to such integrals as $\int \Xi^2(x)\gamma(x)\,dx$ or $\int \Xi(x)\Pi(x)\gamma(x)\,dx$. Nevertheless, operators described with the aid of integrals over squares or products of symbolic operators occur frequently in field quantum theory. Numerical evaluations of the expectation values of such operators have given the value infinity as should have been expected. Therefore it is remarkable that certain operators described in terms of integrals over certain special quadratic forms of $\Xi(x)$ and $\Pi(x)$ can be given a meaning although the product of two symbolic operators is, in general, a meaningless notion. This matter will be discussed later.

The fact that it is possible to give the operators $\Xi(x)$ and $\Pi(x)$ a meaning by associating them with linear forms of operators through formulas (2.6), of course, does not imply that operators exist which satisfy the commutation rule (2.1) or, in other words, that there are pairs of linear forms $\Xi\{\gamma\}$ and $\Pi\{\gamma\}$ such that the identity (2.5) holds. In order to prove the existence of such pairs of operators it appears necessary to exhibit such operators explicitly by representing them with the aid of a functional representation of the states Φ. The major part of this exposition will be concerned with such explicit representations which make it possible to give each mathematical term employed a rigorous definition. On the basis of these definitions it would not be difficult to carry out all mathematical operations considered in a rigorous fashion. Before introducing such representations, however, we shall continue to discuss basic notions of field quantum theory in a formal manner.

3. *The Differential Equations*

So far, the field operators $\Xi(x)$ and the state Φ have tacitly been assumed to refer to a particular time. In order to be able to predict the outcome of future experiments, the change of the operators $\Xi(x)$ and of the state Φ in the course of time must be determined by solving appropriate differential equations.

We shall describe the time changes in the Heisenberg picture. Accordingly, we stipulate that the state Φ does not change in time. We might characterize this state by the outcome of a complete set of measurements made "initially"; clearly, the state so defined does not vary in time since the fact that certain measurements made at a definite instant had certain results remains independent of time. Predictions as to outcomes of future measurements are then to be derived from the knowledge of the change of the operators $\Xi(x) = \Xi(x, t)$ in time.

A typical differential equation for the function $\Xi(x, t)$, which is assumed to hold for electromagnetic or meson fields in case the field does not interact with matter, is the generalized wave equation.

$$(3.1) \qquad (\nabla_t^2 - \nabla^2 + \mu^2)\Xi = 0;$$

see e.g. Wentzel [4, p. 29]. Here

$$\nabla_t = \partial/\partial t, \qquad \nabla^2 = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2 + (\partial/\partial x_3)^2;$$

and μ is a non-negative constant, which is zero for an electromagnetic field and the "meson mass" for a meson field. We have assumed $c = \hbar = 1$; c is the velocity of light and $h = 2\pi\hbar$ is Planck's constant. The solution of the differential equation (3.1) is to be so determined that it satisfies the initial conditions

$$(3.2) \qquad \Xi(x, 0) = \Xi(x),$$

$$(3.3) \qquad \nabla_t\Xi(x, 0) = \Pi(x).$$

$\Xi(x)$ and $\Pi(x)$ are here symbolic operators written as functions of x which satisfy the commutation rule (2.1).

It is clear that the operator $\Xi(x, t)$ has only a symbolic meaning not just for $t = 0$ but for any time t. Nevertheless, one may express the solution of equation (3.1) in terms of the initial values (3.2, 3.3) with the aid of the same formula that gives the solution of the initial value problem for proper functions. To this end let $\Gamma_+(x, t)$ and $\Gamma_-(x, t)$ be the symbolic solutions of the equations

$$(3.4) \qquad (\nabla_t^2 - \nabla^2 + \mu^2)\Gamma = \delta(x)\delta(t)$$

subject to the conditions

$$\Gamma_+(x, t) = 0 \quad \text{for} \quad t < r \tag{3.5}$$

$$\Gamma_-(x, t) = 0 \quad \text{for} \quad t > -r;$$

here

$$r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

Specifically, using the unit step function $\eta(\tau)$ of a real variable τ,

$$\eta(\tau) = \begin{cases} 1 & \text{for} \quad \tau > 0, \\ 0 & \text{for} \quad \tau < 0, \end{cases} \tag{3.6}$$

the solutions Γ_+ and Γ_- can be given explicitly in the form

$$\begin{aligned} \Gamma_+(x, t) &= \frac{\delta(r - t)}{4\pi r} - \eta(t - r)\frac{\mu}{4\pi}\frac{J_1(\mu(t^2 - r^2)^{1/2})}{(t^2 - r^2)^{1/2}}, \\ \Gamma_-(x, t) &= \frac{\delta(r + t)}{4\pi r} - \eta(-t - r)\frac{\mu}{4\pi}\frac{J_1(\mu(t^2 - r^2)^{1/2})}{(t^2 - r^2)^{1/2}}. \end{aligned} \tag{3.7}$$

Here J_1 is the Bessel function of order 1. With the aid of the functions Γ_+ and Γ_- we form the function

$$\tilde{\Gamma}(x, t) = \Gamma_+(x, t) - \Gamma_-(x, t), \tag{3.8}$$

and verify the relations

$$(\nabla_t^2 - \nabla^2 + \mu^2)\tilde{\Gamma}(x, t) = 0, \tag{3.9}$$

$$\tilde{\Gamma}(x, 0) = 0, \tag{3.10}$$

$$\nabla_t\tilde{\Gamma}(x, 0) = \delta(x). \tag{3.11}$$

The function $\tilde{\Gamma}(x, t)$ is generally called "an invariant D-function" associated with equation (3.1) (see e.g. Jordan, Pauli [9] for $\mu = 0$ and Wentzel [4]).

Relations (3.9) and (3.10) can be verified immediately. The verification of relation (3.11) requires a few steps which we shall omit.

From relations (3.9) to (3.11) relation

$$\Xi(x, t) = \int \{\nabla_t\tilde{\Gamma}(x - x', t)\Xi(x') + \tilde{\Gamma}(x - x', t)\Pi(x')\}\, dx' \tag{3.12}$$

could be derived in a standard fashion if $\Xi(x, t)$, $\Xi(x)$ and $\nabla_t\Xi(x, 0) = \Pi(x)$ were proper functions instead of symbolic operators. Relation (3.12) may be considered the symbolic expression of a relation which involves only proper operators. This proper relation is obtained by multiplying relation (3.12) by an arbitrary function $\gamma(x)$ and then employing the expression (2.6). It is not difficult to justify the validity of the resulting proper relation.

An interesting consequence of (3.12) is the commutation rule for the operators $\Xi(x', t')$ and $\Xi(x'', t'')$ at any two space-time points. Using the relations $\Xi(x'')\Xi(x') - \Xi(x')\Xi(x'') = 0$ and

$$\Pi(x')\Xi(x'') - \Xi(x'')\Pi(x') = -i\delta(x'' - x'),$$

we first derive from (3.12) the identities

$$\Xi(x', t')\Xi(x'') - \Xi(x'')\Xi(x', t') = -i\tilde{\Gamma}(x' - x'', t'),$$

$$\Xi(x', t')\Pi(x'') - \Pi(x'')\Xi(x', t') = i\nabla_t\tilde{\Gamma}(x' - x'', t').$$

With the aid of these identities the basic commutation law

$$\Xi(x', t')\Xi(x'', t'') - \Xi(x'', t'')\Xi(x', t') = -i\tilde{\Gamma}(x' - x'', t' - t'') \tag{3.13}$$

may be derived from relation (3.12).

Differentiating this identity with respect to t' and then setting $t' = t'' = t$ we find, incidentally, that the commutation rule

$$\nabla_t\Xi(x', t)\Xi(x'', t) - \Xi(x'', t)\nabla_t\Xi(x', t) = -i\delta(x' - x'') \tag{3.14}$$

holds at any time t. Relations

$$\Xi(x', t)\Xi(x'', t) - \Xi(x'', t)\Xi(x', t) = 0, \tag*{(3.14)$_1$}$$

$$\Pi(x', t)\Pi(x'', t) - \Pi(x'', t)\Pi(x', t) = 0. \tag*{(3.14)$_2$}$$

are verified in a similar way.

Commutation relations of the type (3.13), connecting any pair of points in space-time, were first introduced by Jordan and Pauli [9] in order to formulate commutation rules between field strengths in quantum electrodynamics.

Identity (3.13) exhibits the remarkable fact that the field quantities Ξ at two space-time points commute if they are connected by a space-like line. They do not commute if the two points are connected by a time-like line in case $\mu \neq 0$. In case $\mu = 0$ the operators commute except if the two points are connected by a light ray, $| t' - t'' | = | x' - x'' |$.

Physically, the commutation rule (3.13) for $\mu \neq 0$ means that field quantities which are connected by a time-like line cannot be measured simultaneously, whereas those connected by a space-like line can. In the case of field quantities which are connected by time-like lines, the uncertainty relations obtained for the field quantities at the two points are analogous to Heisenberg's uncertainty relations for momentum and coordinates in the quantum mechanics of particles.

For electromagnetic fields and other fields for which $\mu = 0$, measurements of field strengths at different points cannot be made simultaneously if the points can be connected by a light signal. Uncertainty relations for measurements of field strengths which can be so connected were given by Bohr and Rosenfeld [16].

As is well known, the wave equation is invariant under any Lorentz transformation of the coordinates (x,t) into the coordinates $(\hat{x},\hat{t})$,

$$(\hat{x},\hat{t}) = L(x,t),$$

for which

$$\hat{t}^2 - \hat{r}^2 = t^2 - r^2. \tag{3.15}$$

It is remarkable that also the commutation rules remain invariant; see Pauli, Jordan [9]. Introducing the operators $\hat{\Xi}$ by

$$\hat{\Xi}(\hat{x},\hat{t}) = \Xi(x,t), \tag{3.16}$$

one can infer from the representation (3.13) that the relation

$$\nabla_t \hat{\Xi}(\hat{x}', \hat{t})\hat{\Xi}(\hat{x}'', \hat{t}) - \hat{\Xi}(\hat{x}'', \hat{t})\nabla_t \hat{\Xi}(\hat{x}', \hat{t}) = -i\delta(\hat{x}' - \hat{x}'') \tag{3.17}$$

holds.

Note that the operators Ξ and $\hat{\Xi}$ have only symbolic meaning; the symbolic operator Ξ refers to integration over the x-space, t = constant, while $\hat{\Xi}(\hat{x}, \hat{t})$ refers to integration over the $\hat{x}$-space, $\hat{t}$ = constant. The identification (3.16) would thus be meaningless if it were taken literally. The proper definition of the operator $\hat{\Xi}$ is of course given by defining the linear functional $\int \hat{\Xi}(\hat{x}, \hat{t})\hat{\gamma}(\hat{x})\, d\hat{x}$; this functional can be defined by expressing x and t in terms of $\hat{x}$ and $\hat{t}$ in the right hand member of expression (3.12), multiplying the result by $\hat{\gamma}(\hat{x})$, and integrating with respect to $\hat{x}$.

4. *The Energy Integral*

The solution of the wave equation (3.1) was expressed in terms of the initial data through relation (3.12), in the same way as if the quantity $\Xi(x, t)$ were a proper function. It is a most remarkable fact, to which there is apparently no counterpart with the wave equation for proper functions, that the solution $\Xi(x, t)$ can be expressed in a completely different manner in terms of the initial data. There is a unitary operator $U(t)$, which depends on the initial data $\Xi(x)$ and $\Pi(x)$, such that

$$\Xi(x, t) = U^{-1}(t)\, \Xi(x) U(t). \tag{4.1}$$

Specifically, there is a spectrally representable operator J such that

$$U(t) = \exp\{-itJ\}. \tag{4.2}$$

For many purposes it is more convenient to work with a representation of the operator $\Xi(x, t)$ of the form (4.1), rather than with the expression (3.12).

In order to derive relation (4.1) it is sufficient to verify the differential relation

$$\nabla_t \Xi(x, t) = iJ\Xi(x, t) - i\Xi(x, t)J. \tag{4.3}$$

An operator J which satisfies this relation formally is given by the energy integral. If $\Xi(x, t)$ were a proper function, the integral

$$\tilde{H}(t) = \frac{1}{2}\int \{(\nabla_t\Xi)^2 + (\nabla\Xi)^2 + \mu^2\Xi^2\}\, dx \tag{4.4}$$

would be independent of the time,

$$\tilde{H}(t) = \tilde{H}(0) = \tilde{H}_0 . \tag{4.5}$$

Here

$$\begin{aligned} \Xi &= \Xi(x, t), \\ (\nabla\Xi)^2 &= (\nabla_x\Xi)^2 + (\nabla_y\Xi)^2 + (\nabla_z\Xi)^2, \end{aligned} \tag{4.6}$$

and the integration is to be extended over the whole x-space. The relation

$$\begin{aligned} \frac{d\tilde{H}}{dt} &= \int \{\nabla_t\Xi\nabla_t^2\Xi + \nabla\Xi\nabla_t\nabla\Xi + \mu^2\Xi\nabla_t\Xi\}\, dx \\ &= \int \nabla_t\Xi\{\nabla_t^2\Xi - \nabla^2\Xi + \mu^2\Xi\}\, dx = 0 \end{aligned}$$

is immediately verified if one assumes that the integrals which appear at infinity after integration by parts vanish. It is naturally tempting to introduce the integral $\tilde{H}$ also in the present case in which $\Xi(x, t)$ is a symbolic operator and to adopt the statement that this operator $\tilde{H}$ is constant in time. If this were permitted, remarkable commutation relations would hold. From relation (3.14) we would first derive

$$\begin{aligned} &(\nabla_t\Xi(x', t))^2\Xi(x'', t) - \Xi(x'', t)(\nabla_t\Xi(x', t))^2 = -2i\nabla_t\Xi(x', t)\delta(x' - x''), \\ &(\Xi(x', t))^2\Xi(x'', t) - \Xi(x'', t)(\Xi(x', t))^2 = 0, \\ &(\nabla'\Xi(x', t))^2\Xi(x'', t) - \Xi(x'', t)(\nabla'\Xi(x', t))^2 = 0 \end{aligned} \tag{4.7}$$

and then, from (4.4),

$$\tilde{H}(t)\Xi(x, t) - \tilde{H}(t)\Xi(x, t) = -i\nabla_t\Xi(x, t). \tag{4.8}$$

This relation would be identical with relation (4.3) if one could set

$$J = \tilde{H}(t) = H_0 . \tag{4.9}$$

Thus, if the operator $\tilde{H}$ existed and satisfied the relations (4.5, 4.8), derived formally, the useful representation (4.1) of the operator $\Xi(x, t)$ would be valid.

The integral (4.4), however, is meaningless as it stands; the operators $\nabla_t\, \Xi$ and $\nabla\Xi$ have only a symbolic meaning and hence the operators $(\nabla_t\Xi)^2$ and $(\nabla\Xi)^2$ are meaningless and, indeed, cannot be given any meaning.

Disregarding the fact that the operator $\tilde{H}$ is a priori meaningless, one might, nevertheless, try to find a spectral representation of it. This has been done under the assumption that the field is enclosed in a box; the result was that the operator has a pure point spectrum, each eigenvalue having the value infinity, see e.g. Wentzel [4, p. 35]. Consequently, the only meaning that can be given to the operator H is that it transforms every state into infinity. This result should not be surprising.

What might be surprising is that there is an operator closely related to $\tilde{H}$ which can be given a proper meaning, which is also constant in time, and also satisfies relation (4.2) when taken as operator J. An operator of a similar kind was first introduced by Landau and Peierls in 1930 [12], in a slightly different context.

In order to describe this operator, for which the notation H will be reserved from now on, we must first introduce the square root of the differential operator $-\nabla^2 + \mu^2$. This can be done by using a spectral representation of the operator $-\nabla^2$, as furnished for example by the representation of the functions $\gamma(x)$ through their Fourier transforms. Using the inverse transformation, the result can be expressed as an operator acting on a function $\gamma(x)$. For $\mu = 0$ we thus find

$$(-\nabla^2)^{-1/2}\gamma(x) = (2\pi^2)^{-1} \int | x - x' |^{-2} \gamma(x')\, dx'. \tag{4.10}$$

One may, of course, justify this formula directly. First one should verify the relation

$$(-\nabla^2)^{-1/2}(-\nabla^2)^{-1/2}\gamma(x) = (4\pi)^{-1} \int | x - x' |^{-1} \gamma(x')\, dx' \tag{4.11}$$

or, symbolically, the relation

$$(-\nabla^2)^{-1/2}(-\nabla^2)^{-1/2} = (-\nabla^2)^{-1}, \tag{4.12}$$

in order to make sure that the operator defined by (4.10) is a square root of $(-\nabla^2)^{-1}$. In order to make sure that this operator is the positive definite square root, one should, secondly, show that the operator defined by (4.10) is positive definite. We shall not do this here.

After having introduced the operator $(-\nabla^2)^{-1/2}$ by (4.10) one could define the operator $(-\nabla^2)^{1/2}$ by

$$(-\nabla^2)^{1/2} = (-\nabla^2)^{-1/2}(-\nabla^2). \tag{4.13}$$

The operators so given are to be applied to arbitrary functions $\gamma(x)$ which possess derivatives of a sufficiently high order and vanish sufficiently strongly at infinity.

The operator $(\mu^2 - \nabla^2)^{-1/2}$ can similarly be introduced as an integral operator and then one could define

$$(\mu^2 - \nabla^2)^{1/2} = (\mu^2 - \nabla^2)^{-1/2}(\mu^2 - \nabla^2). \tag{4.14}$$

The appropriate operator H can now be described formally; it is

$$\begin{aligned} H = \frac{1}{2}\int &\{(\mu^2 - \nabla^2)^{1/2}\Xi(x, t) - i\nabla_t\Xi(x, t)\} \\ &\cdot \{(\mu^2 - \nabla^2)^{1/2}\Xi(x, t) + i\nabla_t\Xi(x, t)\}\, dx. \end{aligned} \tag{4.15}$$

This expression involves first the application of the operator $(\mu^2 - \nabla^2)^{1/2}$ and ∇_t on the operators $\Xi(x, t)$ considered as functions of x and t. The meaning of such an application could be explained easily; with reference to operators closely related to $\Xi(x, t)$ this will be done in Section 8, see (8.26). Secondly, the expression (4.15) involves the product of two symbolic operators. Although such a product has a priori no meaning, the operator H, which we shall call the operator of the "intrinsic energy", can be given a precise and proper meaning in a rigorous fashion. This will be done in Section 10.

In a formal manner we shall now compare the operator H with the operator $\tilde{H}$ and prove that H is constant in time.

Carrying out formally the multiplications in the integrand of expression (4.15) we find

$$\begin{aligned} H = \frac{1}{2}\int &[(\mu^2 - \nabla^2)^{1/2}\Xi]^2\, dx + \frac{1}{2}\int (\nabla_t\Xi)^2\, dx \\ &+ \frac{i}{2}\int \{(\mu^2 - \nabla^2)^{1/2}\Xi\nabla_t\Xi - \nabla_t\Xi(\mu^2 - \nabla^2)^{1/2}\Xi\}\, dx. \end{aligned} \tag{4.16}$$

In view of the formulas

$$(4.17) \qquad \int (\mu^2 - \nabla^2)^{1/2}\Xi\cdot\Xi^{(1)}\,dx = \int \Xi\cdot(\mu^2 - \nabla^2)^{1/2}\Xi^{(1)}\,dx,$$

$$\int (\mu^2 - \nabla^2)^{1/2}\Xi\cdot(\mu^2 - \nabla^2)^{1/2}\Xi\,dx = \int \Xi(\mu^2 - \nabla^2)\Xi\,dx,$$

$$-\int \Xi\cdot\nabla^2\Xi\,dx = \int (\nabla\Xi)^2\,dx,$$

we realize that the first two contributions to the expression (4.16) for H are formally just equal to the expression (4.4) for $\tilde{H}$.

From the commutation relations (3.14) one should have

$$(\mu^2 - \nabla^2)^{1/2}\Xi(x, t)\nabla_t\Xi(x, t) - \nabla_t\Xi(x, t)(\mu^2 - \nabla^2)^{1/2}\Xi(x, t)$$
$$= i(\mu^2 - \nabla^2)^{1/2}\delta(x - x) = i(\mu^2 - \nabla^2)^{1/2}\delta(0).$$

If the right hand side here had a meaning, its value could only be infinite. Nevertheless, one may say that the formal expression obtained for it is independent both of the choice of the operators $\Xi(x, t)$ and of time. Accordingly, one is inclined to say that the last contribution to the expression (4.16) for H is an infinite constant, the so-called "zero-point energy". Since, as was stated above, the expression $\tilde{H}$ is also infinite, one might say that the operator H, shown to be finite in Section 10, represents the finite part of the energy operator $\tilde{H}$.

The operator H is occasionally mentioned in the quantum physical literature, but its importance is not emphasized strongly. One of the doubts that had been raised concerning the idea that the operator H, instead of $\tilde{H}$, should represent the total energy, stemmed from the fact that then the energy density, i.e. the integrand in the expression (4.15) would no longer depend only on the local values of the field quantity Ξ and its derivatives. This objection is not quite convincing; after all, more fundamental features of classical physics had to be abandoned in quantum theory. Certainly, the two fundamental properties of the energy are preserved, when H is adopted as energy operator J, viz., its constancy in time and its role in the formation of the unitary operator $U(t)$ which governs the time change of the field operator Ξ.

In a formal manner it is easy to prove the constancy of the operator H directly. Indeed, using the differential equation (3.1) and identity (4.17) we have

$$\frac{dH}{dt} = \frac{1}{2}i \int [(\mu^2 - \nabla^2)^{1/2}\{-i\nabla_t\Xi + (\mu^2 - \nabla^2)^{1/2}\Xi\}\cdot\{(\mu^2 - \nabla^2)^{1/2}\Xi + i\nabla_t\Xi\}$$
$$- \{(\mu^2 - \nabla^2)^{1/2}\Xi - i\nabla_t\Xi\}\cdot(\mu^2 - \nabla^2)^{1/2}\{i\nabla_t\Xi + (\mu^2 - \nabla^2)^{1/2}\}\Xi]\,dx = 0$$

Relation (4.3) is also easily verified; for, the relation

$$\{(\mu^2 - \nabla^2)^{1/2}\Xi(x', t) \mp i\nabla_t\Xi(x', t)\}\Xi(x'', t) - \Xi(x'', t)\{(\mu^2 - \nabla^2)^{1/2}\Xi(x', t) \mp i\nabla_t\Xi(x', t)\} = \mp\delta(x' - x''),$$

which follows from (3.14), entails the relation

$$\begin{aligned}\{(\mu^2 - \nabla^2)^{1/2}\Xi(x', t) - i\nabla_t\Xi(x', t)\}&\\ \cdot\{(\mu^2 - \nabla^2)^{1/2}\Xi(x', t) + i\nabla_t\Xi(x', t)\}\Xi(x'', t)&\\ - \Xi(x'', t)\{(\mu^2 - \nabla^2)^{1/2}\Xi(x', t) - i\nabla_t\Xi(x', t)\}&\\ \cdot\{(\mu^2 - \nabla^2)^{1/2}\Xi(x', t) + i\nabla_t\Xi(x', t)\}&\\ = -2i\nabla_t\Xi(x', t)\delta(x' - x''),&\end{aligned}$$

and hence (4.3).

5. *Motivation of the Configuration Space Representation*

Operators related to the operators $(\mu^2 - \nabla^2)^{1/2}\Xi(x) \mp i\Pi(x)$, which enter the definition (4.15) of the energy operator H, will play a basic role in the subsequent sections. In particular, we shall employ the operators

$$A^{\pm}(x) = 2^{-1/2}[(\mu^2 - \nabla^2)^{1/4}\Xi(x) \mp i(\mu^2 - \nabla^2)^{-1/4}\Pi(x)]. \tag{5.1}$$

One of the advantages in working with these operators results from the fact that they satisfy the simple commutation laws

$$A^-(x')A^+(x'') - A^+(x'')A^-(x') = \delta(x' - x''), \tag{5.2}$$

$$A^+(x')A^+(x'') - A^+(x'')A^+(x') = 0, \tag{5.2$^+$}$$

$$A^-(x')A^-(x'') - A^-(x'')A^-(x') = 0, \tag{5.2$^-$}$$

which follow immediately from relations (2.1). With the aid of the operators $A^{\pm}(x)$ the operator H can be written in the form

$$H = \int A^+(x)(\mu^2 - \nabla^2)^{1/2}A^-(x)\,dx. \tag{5.3}$$

For many considerations it is more suitable to work with the operator

$$N = \int A^+(x)A^-(x)\,dx. \tag{5.4}$$

In fact, by investigating the properties of this operator N one is naturally led to the representation of the states Φ which will be the basis for the considerations of the subsequent sections; see Becker and Leibfried [15]. First we note that the operators $A^+(x)$ and $A^-(x)$ are conjugate to each other so that the operators $A^+(x)A^-(x)$ are non-negative. The same is therefore true of the operator N. No eigenvalue of this operator can therefore be negative.

Now we assume that there is an eigenstate Φ of the operator N with the eigenvalue η so that

$$N\Phi = \eta\Phi. \tag{5.5}$$

With the aid of this state Φ we shall construct an eigenstate of N with the eigenvalue zero. Suppose Φ itself is not such an eigenstate. Then we form the state

$$\Phi^{(1)} = A^-(x_1)\Phi. \tag{5.6}$$

Certainly, the value of x_1 can be so chosen that $\Phi^{(1)} \neq 0$; for, if $A^-(x)\Phi = 0$ for all values of x, also $N\Phi = \int A^+(x)A^-(x)\, d\Phi = 0$, contrary to assumption.

To be precise we should have defined the state $\Phi^{(1)}$ by

$$\Phi^{(1)} = \int g_1(x)A^-(x)\, dx\, \Phi, \tag{5.7}$$

with an appropriate function $g_1(x)$; for, $A^-(x_1)$ is not a proper operator. We then should have shown that $N\Phi = 0$ if these states $\Phi^{(1)}$ vanish for all choices of the function $g_1(x)$. Since the present consideration is only heuristic, we shall not carry this through.

We now apply the operator N on the state $\Phi^{(1)}$ defined by (5.6) and deduce from the commutation rule (5.2) the relation

$$N\Phi^{(1)} = (\eta - 1)\Phi^{(1)}, \tag{5.8}$$

which implies that the state $\Phi^{(1)}$ is an eigenstate of N with the eigenvalue $\eta - 1$. Continuing the argument we find a sequence of eigenstates $\Phi^{(\nu)}$ with the eigenvalues, $\eta - \nu$, $\nu = 1, 2, \cdots$. This sequence can end only if one of these eigenvalues is zero. If that were never the case, negative eigenvalues would occur, which is impossible as mentioned above. Therefore, the existence of an eigenstate $\hat{\Phi}$ with the eigenvalue zero is deduced from the assumption that some eigenstate exists.

From one such eigenstate $\hat{\Phi}$ with an eigenvalue zero a manifold of eigenstates with the eigenvalue n can be constructed by a simple process in which the operators $A^+(x)$ are used. To this end one chooses functions $\psi_n(x_1, \cdots, x_n)$ of n variables $x_1, \cdots, x_n$ and forms the state

$$\Phi_n = \int \cdots \int \psi_n(x_1, \cdots, x_n)A^+(x_1) \cdots A^+(x_n)\, dx_1 \cdots dx_n\, \hat{\Phi}, \tag{5.9}$$

$$n > 0, \qquad \Phi_0 = \psi_0\hat{\Phi}, \qquad n = 0, \qquad \psi_0 \text{ being any number.}$$

Because of relation $(5.2)^+$ no generality is lost by assuming the function $\psi_n(x_1, \cdots, x_n)$ to be symmetric in $x_1, \cdots, x_n$; i.e. to be invariant under all permutations of these variables.

From the commutation rules (5.2) one easily derives the relation

$$N\Phi_n = n\Phi_n, \tag{5.10}$$

which shows that Φ_n is an eigenfunction of N with the eigenvalue n.

Evidently, the state Φ_n is completely characterized and, more specifically, represented by the function $\psi_n(x_1, \cdots, x_n)$.

Quite generally, any state Φ which can be written as a superposition

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots \tag{5.11}$$

of states of the type (5.9), is represented by the number ψ_0 and the sequence of functions $\psi_1(x_1)$, $\psi_2(x_1, x_2)$, $\cdots$. We express this fact by writing

$$\Phi \leftrightarrow \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \cdots\}. \tag{5.12}$$

This is the "configuration space" representation, given by Landau and Peierls [12] and by Fock [13], which will be the basis of the considerations in the subsequent sections; it will later on be reached from a different starting point.

It may be mentioned that a representation of the form (5.12) could also be deduced in a similar fashion if the commutation relations (5.2), $(5.2)^{+-}$ are modified by substituting the plus sign for the minus sign in their left members. The only difference would be that without loss of generality the functions $\psi_n(x_1, \cdots, x_n)$ may be assumed as antisymmetric, i.e. as invariant under odd permutations.

The discussion in this section is, of course, only heuristic and serves mainly to motivate the notions introduced later on. Nevertheless it seems likely that this discussion could be made rigorous by employing arguments of the type Rellich [21] used to prove rigorously the essential uniqueness of operators p and q which satisfy the commutation law $pq - qp = i$. By such arguments it should also be possible to justify the assumption on which the present discussion is based, that the operator N has only point eigenvalues if this operator is defined at all. At the same time, it should be possible to establish the essential uniqueness of the representation (5.12).

The assumption that the operator N is defined was essential in the preceding discussion. As will be shown in Part IV, there are other realizations of operators A^* satisfying the commutation rules (5.2) in which the operator N is not defined. In other words, there are different realizations of field operators Ξ and Π which satisfy the commutation rules (2.1). However, only in the realization in which the operator N is defined is it possible to describe the states of the field in terms of particles. It seems that this is the realization that corresponds to most experimental situations.

Part II. Particle Representation

6. *Biquantization*

The explicit representation of the field operators, such as $\Xi(x)$, $\Pi(x)$, and H, to which we have referred in the previous section will be given in connection with a functional representation of the states which, in particular, can be set up so as to yield a spectral representation of the energy operator H. It will be seen that this representation can be described in terms of the spectral representation of an operator N, already mentioned in Section 5, which commutes with H, and which has only the point eigenvalues $n = 0, 1, 2, \cdots$. If the field is in an eigenstate of this operator N, associated with the eigenvalue n, one may say that the field consists of n particles. Accordingly, we shall also speak of a particle representation.

A natural approach to the particle representation, different from the approach described in Section 5, consists in starting with the quantum theory of a single particle and then developing the process of second quantization or, as we prefer to say, of "biquantization".

In the following discussion we make extensive use of the notions of functional and spectral representation which were indicated at the end of the introduction; we shall slightly modify the notation in order to adapt it to the specific needs of our present problems.

We denote by S any complete system of commuting operators which correspond to observables associated with the particle. The eigenvalues of the operators S will be denoted by s. Of course, the symbol s here stands for a finite set of numbers, just as the symbol S stands for a finite set of operators. The state Ψ of a single particle may then be represented by a function $\psi(s)$. In order to complete the functional representation an appropriate measure function $m(\mathfrak{S})$ of sets $\mathfrak{S}$ in the s-space must be given so that the manifold of the representers of the states Ψ consists just of those functions $\psi(s)$ for which

$$\int |\psi(s)|^2 \, dm(s) < \infty. \tag{6.1}$$

The Hilbert state of states Ψ will be denoted by $\mathfrak{H}$ and the manifold of representers $\psi(s)$ by $\mathfrak{R}$.

The inner product of two states Ψ and $\Psi^{(1)}$ in $\mathfrak{H}$ is expressed in terms of these representers $\psi(s)$ and $\psi^{(1)}(s)$ by

$$(\Psi, \Psi^{(1)}) = \int \overline{\psi(s)}\psi^{(1)}(s) \, dm(s). \tag{6.2}$$

Of course, the function $\psi(s)$ is a probability amplitude. More specifically, the probability that a measurement of the observables S will yield a value s in a set $\mathfrak{S}$ is just $\int_{\mathfrak{S}} |\psi(s)|^2 \, dm(s)$.

The system of observables S may, for example, consist of the position $X = \{X_1, X_2, X_3\}$ or of the momentum $K = \{K_1, K_2, K_3\}$. In that case we speak of a position or X-representation, or of a momentum or K-representation respectively.

For a field enclosed in a box the variables s may be chosen to be any system of "quantum numbers"; here a quantum number is any label used to characterize the eigenstates of the energy.

We shall find it convenient to call the variable s the "quantum variable" whether it stands for some physically measurable quantity such as position or momentum or is used as a label.

Let Λ be any operator acting on states Ψ of the single particle. We assume this operator to be Hermitian and, moreover, to be spectrally representable (see the end of the introduction) in a dense subspace $\mathfrak{G}'$ of $\mathfrak{G}$. The subspace of $\mathfrak{K}$ corresponding to $\mathfrak{G}$ will be denoted by $\mathfrak{K}'$.

Corresponding to the representation of the states Ψ in $\mathfrak{G}$ by functions $\psi(s)$ in $\mathfrak{K}$, the operator Λ is represented by a functional operator acting on functions in $\mathfrak{K}'$. As explained at the end of the introduction, we find it convenient for our purposes to denote this functional operator simply by Λ^S so that the representation

$$\Psi \leftrightarrow \psi(s) \tag{6.3}$$

of states Ψ in $\mathfrak{G}$ by functions in $\mathfrak{K}$ entails the representation

$$\Lambda\psi \leftrightarrow \Lambda^S\psi(s) \tag{6.4}$$

for Ψ in $\mathfrak{G}'$ and $\psi(s)$ in $\mathfrak{K}'$; we called the functional operator Λ^S the representer of the operator Λ. On occasion we find it convenient to write Λ^s instead of Λ^S.

Of particular importance is the case that the operator is a function of the operators S,

$$\Lambda = \lambda(S). \tag{6.5}$$

In that case the state $\Lambda\Psi$ is simply represented by the function $\lambda(s)\psi(s)$,

$$\lambda(S)\Psi \leftrightarrow \lambda(s)\psi(s). \tag{6.6}$$

The space $\mathfrak{K}'$ can then be precisely characterized as the space of all those functions $\psi(s)$ in $\mathfrak{K}$, for which

$$\int |\lambda(s)|^2 \, |\psi(s)|^2 \, dm(s) < \infty. \tag{6.7}$$

The representation $\Psi \leftrightarrow \psi(s)$ may be said to yield a spectral representation of the operator Λ. In fact, the functions

$$\psi(s) = c, \qquad c = \text{const.} \qquad \text{for} \quad s = s', \tag{6.8}$$
$$\psi(s) = 0, \qquad\qquad\qquad \text{for} \quad s \neq s',$$

are then eigenfunctions of the operator Λ^S with the eigenvalue $\lambda(s')$, since they satisfy the relation

$$\Lambda^S\psi(s) = \lambda(s')\psi(s) \tag{6.9}$$

as seen from (6.4, 6.6, 6.8). The eigenfunction (6.8) may be proper or improper. It is proper if the measure $m(\mathcal{S})$ of the set $\mathcal{S}$ differs from zero when the set $\mathcal{S}$ consists only of the point $s = s'$; we then say, $m(\mathcal{S})$ has a "jump" at this point. In that case, the function (6.8) belongs to the space $\mathfrak{K}$ and represents a state in $\mathfrak{G}$. If $m(\mathcal{S})$ has no jump at s', the function (6.8) does not belong to $\mathfrak{K}$ and hence does not represent a state in $\mathfrak{G}$. Incidentally, all eigenfunctions of Λ^S belonging to the eigenvalue λ are linear combinations of the eigenfunctions (6.8) for those values of s' for which $\lambda(s') = \lambda$.

In order to exemplify the notions just discussed, let us consider the momentum operator K. Its K-representation, of course, is $K^K = k$, and its X-representation is

$$K^X = -i\nabla = -i\frac{\partial}{\partial x} \tag{6.10}$$

here $\nabla = \partial/\partial x$ is the gradient. Properly the momentum is $P = \hbar K$; but we have assumed $\hbar = 1$, see Section 3. The operator corresponding to the kinetic energy of the particle will be denoted by Ω. In general, we shall assume that Ω is given by Einstein's expression $(\mu^2c^2 + P^2)^{1/2}c$ or, because of $c = \hbar = 1$, by

$$\Omega = \omega(K) = (\mu^2 + |K|^2)^{1/2}, \tag{6.11}$$

see Section 2, so that

$$\Omega^K = (\mu^2 + |k|^2)^{1/2} \tag{6.12}$$

and

$$\Omega^X = (\mu^2 - \nabla^2)^{1/2}. \tag{6.13}$$

Here ∇^2 is the Laplacian differential operator and μ is the mass of the particle. This energy operator was adopted by Yukawa in his meson theory; it also corresponds to the absolute value of the energy of electrons and positrons in Dirac's theory, (except that neither X nor K forms a complete set of operators in his theory). Also the energy operator of photons is closely related to expressions (6.11); in this case $\mu = 0$ however.

The time dependence of the state $\Psi(t)$ in the Schrödinger picture, in which the operators X, K, Ω do not vary in time, is given by

$$\Psi(t) = \exp\{-it\Omega\}\Psi(0), \tag{6.14}$$

so that the time dependence of the representer is given by

$$\psi(s, t) = \exp\{-it\Omega^S\}\psi(s, 0). \tag{6.15}$$

In a momentum representation this expression becomes

$$\psi(k, t) = \exp\{-it\omega(k)\}\psi(k, 0). \tag{6.16}$$

If a momentum eigenfunction $\psi(k, 0) = c\delta(k - k')$ is chosen the function

$$\psi(k, t) = \exp\{-it\omega'\}c\delta(k - k') \tag{6.17}$$

represents a "stationary", though improper, state. Here $\omega' = \omega(k') = (\mu^2 + |k'|^2)^{1/2}$. If the function $\psi(k, 0)$ is quadratically integrable and vanishes for $|k - k'| \leq \epsilon$, the function $\psi(k, t)$ represents a proper, but only approximately stationary, state.

The next step to be taken, before biquantization can be introduced, is to consider an *assembly of n particles.* Such an assembly may be in a state Ψ_n represented by a function $\psi_n(s_1, \cdots, s_n)$ for which

$$\int \cdots \int |\psi_n(s_1, \cdots, s_n)|^2 \, dm(s_1) \cdots dm(s_n) < \infty. \tag{6.18}$$

The manifold of such functions will be denoted by $\mathfrak{R}_n$ and the manifold of states Ψ_n so represented by $\mathfrak{G}_n$. For convenience we use the abbreviating notations

$$(s_1, \cdots, s_n) = (s)_n \cdot \tag{6.19}$$

$$dm(s_1) \cdots dm(s_n) = dm(s)_n .$$

Accordingly, we set

$$\psi_n(s_1, \cdots, s_n) = \psi_n(s)_n \tag{6.20}$$

and write condition (6.18) in the form

$$\int |\psi_n(s)_n|^2 \, dm(s)_n < \infty. \tag{6.21}$$

The inner product of two states Ψ_n, $\Psi_n^{(1)}$ with the representers $\psi_n(s)_n$ and $\psi_n^{(1)}(s)_n$ is then given by

$$(\Psi_n, \Psi_n^{(1)}) = \int \bar{\psi}_n(s)_n \psi_n^{(1)}(s)_n \, dm(s)_n . \tag{6.22}$$

Any product

$$\psi_n(s)_n = \psi^{(1)}(s_1) \cdots \psi^{(n)}(s_n) \tag{6.23}$$

of n functions in $\mathfrak{R}$ evidently forms a function in $\mathfrak{R}_n$. If the state Ψ_n of the assembly is represented by such a product one may say that the first particle is in the state $\Psi^{(1)}$ represented by $\psi^{(1)}(s)$, the second in the state $\Psi^{(2)}$ represented by $\psi^{(2)}(s)$, and so on. Incidentally, the space $\mathfrak{K}_n$ can be shown to be the closure of the linear combinations of products (6.23). Of course, $\mathfrak{K}_1 = \mathfrak{K}$.

Let Λ be a spectrally representable operator defined in a subspace $\mathfrak{S}'$ of $\mathfrak{S}$ and Λ^S its representer defined in the subspace $\mathfrak{K}'$ of $\mathfrak{K}$. We shall introduce a dense subspace $\mathfrak{K}_n'$ of the space $\mathfrak{K}_n$ of functions $\psi_n(s)_n$ and define an operator, written as

$$\Sigma^n\Lambda^S = \Lambda^{S_1} + \cdots + \Lambda^{S_n}, \tag{6.24}$$

which is applicable to these functions. To define this operator and the space $\mathfrak{K}_n'$ consider first n functions $\psi^{(1)}(s), \cdots, \psi^{(n)}(s)$ in $\mathfrak{K}'$. Then the product $\psi_n(s)_n = \psi^{(1)}(s_1) \cdots \psi^{(n)}(s_n)$ admits the operator $\Sigma^n\Lambda^S$ defined by

$$\Sigma^n\Lambda^S\psi_n(s)_n = (\Lambda^{S_1}\psi^{(1)}(s_1)) \cdots \psi^{(n)}(s_n) + \cdots + \psi^{(1)}(s_1) \cdots (\Lambda^{S_n}\psi^{(n)}(s_n)).$$

Next we define the operator for all linear combinations of such products, and, finally, the space $\mathfrak{K}_n'$ and in it the operator $\Sigma^n\Lambda^S$ through the process of closure; for this notion see v. Neumann [19, p. 75, 76]. Without proof we state that the operator $\Sigma^n\Lambda^S$ so defined in the space $\mathfrak{K}_n'$ is spectrally representable.

Of particular interest is the case that the operator Λ is a function of the complete system of operators S, $\Lambda = \lambda(S)$. Then the space $\mathfrak{K}_n'$ consists just of those functions $\psi_n(s)_n$ for which

$$\int |\, \Sigma^n\lambda(s) \,|^2 \, |\, \psi_n(s)_n \,|^2 \, dm(s)_n < \infty, \tag{6.25}$$

and the operator $\Sigma^n\Lambda^S$ transforms $\psi_n(s)_n$ into

$$\Sigma^n\Lambda^S\psi_n(s)_n = \Sigma^n\lambda(s)\psi_n(s)_n = (\lambda(s_1) + \cdots + \lambda(s_n))\psi_n(s_1, \cdots, s_n). \tag{6.26}$$

As a special case we mention the operator of the total energy of an assembly of n particles. It is represented by the operator $\Sigma^n\Omega^S$. If $S = K$ and $\Omega = \omega(K)$, this operator consists just in multiplying the function $\psi_n(k_1, \cdots, k_n)$ by $\omega(k_1) + \cdots + \omega(k_n)$. It should be noted that the term total energy here refers to kinetic energy only and does not include the energy of possible interaction between the particles.

The process of second quantization or *biquantization* can now be formulated. It consists in introducing a new physical entity, called "field", whose states Φ may be represented by a system of functions $\psi_n(s)_n$ in $\mathfrak{K}_n$ and a number ψ_0,

$$\Phi \leftrightarrow \{\psi_0, \psi_1(s_1), \psi_2(s)_2, \cdots\}, \tag{6.27}$$

for which

$$|\, \psi_0 \,|^2 + \int |\, \psi_1(s_1) \,|^2 \, dm(s_1) + \int |\, \psi_2(s)_2 \,|^2 \, dm(s)_2 + \cdots < \infty. \tag{6.28}$$

If the number ψ_0 and all the functions $\psi_1(s_1)$, $\cdots$ except $\psi_n(s)_n$ vanish we say the field is in a state in which it consists of an assembly of n particles; $\psi_n(s)_n$ is then the amplitude of the probability that for these particles the observables S will be found to have the values $s_1, \cdots, s_n$. If all functions $\psi_n(s)_n$, $n = 1, 2, \cdots$ vanish, but $\psi_0 \neq 0$, we say the field consists of no particle, or the field is in the *vacuum* state.

In the following it is convenient to admit the value 0 for n. Accordingly, we define $\psi_0(s)_0$ as the number

$$\psi_0(s)_0 = \psi_0 \tag{6.29$_1$}$$

and further

$$\int |\psi_0(s)_0|^2 \, dm(s)_0 = |\psi_0|^2. \tag{6.29$_2$}$$

We then write the representation (6.27) of the state Φ as

$$\Phi \leftrightarrow \{\psi_n(s)_n\} \tag{6.30}$$

and give the inner product of Φ and $\Phi^{(1)}$ by

$$(\Phi, \Phi^{(1)}) = \sum\nolimits_n \int \overline{\psi_n}(s)_n \psi_n^{(1)}(s)_n \, dm(s)_n \,. \tag{6.31}$$

The Hilbert space of all the states Φ so represented will be denoted by $\mathfrak{H}$.

The state Φ_n with the representation

$$\Phi_n \leftrightarrow \{\cdots 0, \psi_n(s)_n, 0 \cdots\}, \tag{6.32}$$

will also be called the n-th component of the state Φ.

The representation (6.30) is certainly a functional representation of the type described at the end of the introduction. The independent variables are $n, s_1, s_2, \cdots$. In order to describe the measure of a set $\mathbb{S}$ in the space of these infinitely many variables, let $\mathbb{S}_\nu$ be the intersection of the set $\mathbb{S}$ with the manifold $s_{\nu+1} = s_{\nu+2} = \cdots = 0$, and let $\delta(n - \nu) = 1$ for $n = \nu$, $=0$ for $n \neq \nu$. Then the measure of $\mathbb{S}$ is

$$m(\mathbb{S}) = \delta(n - 0)m(\mathbb{S}_0) + \delta(n - 1)m(\mathbb{S}_1) + \cdots.$$

Here $m(\mathbb{S}_0) = 0$ if $\mathbb{S}_0$ is empty, $=1$ if $\mathbb{S}_0$ is the point $s_1 = s_2 = \cdots = 0$.

There could be no doubt how one should define the operator which acts on the states of the field corresponding to an operator Λ which acts on the states Ψ of a single particle belonging to a subspace $\mathfrak{S}'$ of $\mathfrak{S}$. Let $\mathfrak{H}'$ be the subspace of $\mathfrak{H}$ consisting of those states whose representers $\psi_n(s)_n$ belong to the states $\mathfrak{K}'_n$ and for which

$$\sum\nolimits_n \int |\Sigma^n \Lambda^S \psi_n(s)_n|^2 \, dm(s)_n < \infty; \tag{6.33}$$

here the summation runs over 0, 1, 2, $\cdots$; by $\Sigma^0 \Lambda^S$ we mean

$$\Sigma^0 \Lambda^S = 0. \tag{6.34}$$

The operator $[\Lambda]$ is defined as acting on the states Φ in $\mathfrak{H}'$ through the representation

$$[\Lambda]\Phi \leftrightarrow \{\Sigma^n \Lambda^S \psi_n(s)_n\}. \tag{6.35}$$

The operator $[\Lambda]$ so defined is said to result from *biquantization* of the operator Λ. For appropriate functions $f(\lambda)$ of a variable λ and appropriate states Φ the operator $f([\Lambda])$ can be defined by

$$f([\Lambda])\Phi \leftrightarrow \{f(\Sigma^n \Lambda^S)\psi_n(s)_n\}. \tag{6.36}$$

For example, if the energy operator Ω is a function $\omega(K)$ of K, the energy of the field is defined by

$$[\Omega]\Phi \leftrightarrow \{0,\ \omega(k_1)\psi_1(k_1),\ (\omega(k_1) + \omega(k_2))\psi_2(k_1\,, k_2),\ \cdots\}. \tag{6.37}$$

In particular, the vacuum state,

$$\Phi_0 \leftrightarrow \{1, 0, \cdots\}$$

is transformed into

$$[\Omega]\Phi_0 = 0;$$

in other words in the vacuum state the field has the energy zero.

We further note that the representation of the time dependent Schrödinger state $\Phi(t)$ is given by

$$\Phi(t) = \exp\{-it[\Omega]\}\Phi(0) \leftrightarrow \{\exp\{-it\Sigma^n\omega(k)\}\psi_n(k)_n\}. \tag{6.38}$$

It should be recalled that the operator $[\Omega]$ can be adopted as the correct energy operator only when the particles do not interact with each other.

The operator [1] resulting from biquantization of the identity is called N,

$$N = [1]; \tag{6.39}$$

this operator is evidently represented by

$$N\Phi \leftrightarrow \{0,\ \psi_1(s_1),\ 2\psi_2(s)_2\,,\ \cdots\} = \{n\psi_n(s)_n\}. \tag{6.40}$$

The particle representation (6.30) therefore yields a spectral representation of this operator N with the eigenvalues $n = 0, 1, \cdots$. The particle representation will also be called an N-representation or, still more specifically, an (N,S)-representation.

The (N,X)-representation is apparently the same representation to which we were led by the discussion of Section 5, see (5.12). The (N,X)-representation is frequently referred to as the representation in the "configuration space". It is suggested in the early works of Dirac and Heisenberg but apparently first mentioned explicitly by Landau and Peierls in (1930) [12]; it was discussed in detail as the basis of a representation of fields and of second quantization by Fock [13].

We must discuss an important additional restriction on this representation. It appears that in nature assemblies of particles and fields do not occur in states having all possible N-representations of the type (6.30); they occur only in states having N-representers belonging to two separate manifolds.

The first manifold of these representers consists of functions $\psi_n(s_1, \cdots, s_n)$ which are symmetric in the variables $s_1, \cdots, s_n$; i.e. they are invariant under any permutation of the order of the variables $s_1, \cdots, s_n$. Particles are called "bosons" by Dirac if any assembly composed of similar such particles can be only in states represented by symmetrical representers. Fields of an undetermined number of such particles will be called boson fields.

The second manifold of representers of the states of an assembly of n particles $\psi_n(s_1, \cdots, s_n)$ consists of functions which are anti-symmetric in the variables $s_1, \cdots, s_n$; i.e. they just change sign under an odd permutation of the order of the variables $s_1, \cdots, s_n$ and are hence invariant under any even permutation. If assemblies of similar particles occur only in states represented by anti-symmetric representers, then Dirac calls the particles fermions. Anti-symmetric functions can never be of the form $\psi_n(s_1, \cdots, s_n) = \psi(s_1)\psi(s_2) \cdot \psi_{n-2}(s_3, \cdots, s_n)$, which involves the product of the same function of s_1 and s_2. In other words, the state of the assembly of n particles can never be such that it implies that two particles are in the same state. This fact is Pauli's "exclusion principle". Dirac [6] and Heisenberg [7] were the first to show the significance of the symmetric and antisymmetric representers.

States represented by symmetric or anti-symmetric functions will themselves be called symmetric and anti-symmetric respectively. The two types of states leading to the two types of particles will play a major role in the discussion of the subsequent sections. It is to be noted that the absolute values of the representers for an assembly of bosons or fermions remain unchanged when any two arguments are interchanged. The individual particles are thus indistinguishable.

7. *Remark on the Occupation Number Representation*

The process of biquantization in terms of a particle representation is often described in the literature in a different manner, first treated in detail by Dirac [8] for bosons and by Jordan and Wigner [10] for fermions. We stipulate for this description that the spectrum of the energy operator be discrete. This can be brought about by assuming the field to be enclosed in a box. It is then natural to choose the operator S such that it also has a discrete spectrum consisting of simple eigenvalues s^r, $r = 1, 2, \cdots$, and that, in addition, the energy is a function of S. The representers $\psi_n(s_1, \cdots, s_n)$ can then be described by an array of numbers $\psi_n(s^{r_1}, \cdots, s^{r_n})$; these numbers may be considered the coefficients of the expansion of the state ψ with respect to eigenstates of the system of operators S. Each such eigenstate is characterized by the set of numbers $s^{r_1}, \cdots, s^{r_n}$. Assuming that the function ψ_n is either symmetric or anti-symmetric, an eigenstate can be characterized solely by indicating, for every eigenvalue s^r, the number n_r of those values in the set $s^{r_1}, s^{r_2}, \cdots, s^{r_n}$ which equal s^r. This fact is frequently expressed by saying that n_r is the number of particles which are in the state (r) characterized by the eigenvalue s^r; the number n_r is therefore called the "occupation number". The representer $\psi_n(s_n)$ depends only on the occupation numbers, $\psi_n(s_n) = \psi'(n_1, n_2, \cdots)$ with $n_1 + n_2 + \cdots = n$. Frequently, one also introduces the operator N_r which multiplies every eigenstate of S by the occupation number n_r. Evidently, $\sum_r N_r = N$, i.e. the sum of all operators N_r is the operator N, whose eigenvalue in an eigenstate of S is the total number n of particles. From this point of view the operators N_r are spectrally represented, if the state Φ is represented by the set of function ψ',

$$\Phi \leftrightarrow \{\psi'(n_1, n_2, \cdots)\} = \{\psi'(n_r)\}.$$

This representation is quite different from the N-representation

$$\Phi \leftrightarrow \{\psi_n(s_1, s_2, \cdots, s_n)\} = \{\psi_n(s)_n\}.$$

In many descriptions of field theory a box is introduced artificially and then the somewhat more involved occupation number representation is employed; finally a limiting process is carried out by letting the box expand indefinitely. Such a procedure appears to be an unnecessary detour. For the problems of scattering and others of a similar type with which we are primarily concerned here, it is preferable to use a description, such as the particle description, which can immediately be given for the infinite space.

Nevertheless, there are properties of a field whose formulation requires just the occupation number representation.[4] This is true of thermodynamic properties which are to be derived from the statistics of the eigenstates of the energy. Such eigenstates exist only if the field is enclosed in a box and, indeed, a thermodynamic notion such as the pressure, for example, requires for its definition reference to the interaction between the field and a finite enclosure.

However, the occupation number representation is not necessarily associated with the "box". In fact, an occupation number representation can also be set up if the spectrum of the operator S is not discrete. In particular the states of fields in infinite space can be described in terms of occupation numbers. This will be shown in Section 16. Such a description, however, involves peculiar limit processes which are not very easily handled analytically.

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[4] I owe the subsequent observation to H. E. Moses.

It should also be mentioned that there are occupation number representations which are not equivalent with a particle representation, because no operator N can be associated with them. This fact, alluded to at the end of Section 5, will also be discussed in Part IV.

8. *Annihilation and Creation Operators*

Suppose the state of the field, in the Schrödinger picture, varies in the course of time in such a way that every eigenstate of the number operator N remains such an eigenstate. In that case the number n of particles in the field would not change and it would hardly be worthwhile to carry out the process of second quantization. The second quantization is, however, a necessity if one wants to describe situations in which a state which originally was an eigenstate of the operator N does not remain such an eigenstate in the course of time. Such a situation may come about due to the effect of external forces which we have not yet considered. There is a non-vanishing probability that a measurement of the number of particles in the field at a later time will yield a value different from the original one. In other words, there will be a probability that particles have been created or annihilated.

In order to describe such transitions it is useful to introduce operators which transform an eigenstate of the operator N, associated with the eigenvalue n, into an eigenstate associated with the value $n - 1$ or $n + 1$ respectively. Such operators are called annihilation and creation operators, respectively.

In particular, following Fock, we introduce an *annihilation operator* $A^-(s')$, assigned to an arbitrary value of the parameter s', by stipulating that it transforms the state Φ, represented by (6.30) through

$$(8.1) \qquad \Phi \leftrightarrow \{\psi_0 , \psi_1(s_1), \cdots , \psi_n(s_1 , \cdots , s_{n-1} , s_n), \cdots\} = \{\psi_n(s)_n\}$$

into the state $A^-(s')\Phi$, represented by

$$A^-(s')\Phi$$

$$(8.2) \qquad \leftrightarrow \{\psi_1(s'), 2^{1/2}\psi_2(s_1 , s'), \cdots , (n + 1)^{1/2}\psi_{n+1}(s_1 , \cdots , s_n , s'), \cdots\}$$

$$= \{(n + 1)^{1/2}\psi_{n+1}(s)_n(s')\}.$$

Here the quantity $\psi_{n+1}(s)_n(s')$ is defined by

$$\psi_{n+1}(s)_n(s') = \psi_{n+1}(s_1 , s_2 , \cdots , s_n , s').$$

The reason for inserting the square roots will become apparent later on.

It is to be noted that the operator $A^-(s')$ preserves the symmetry properties of the representers for boson and fermion fields; that is to say, if the state Φ has as its representer a symmetric or anti-symmetric function, the state $A^-(s')\Phi$ will have as its representer a symmetric or anti-symmetric function respectively. The vacuum state, $\Phi_0 \leftrightarrow \{1, 0, 0, 0, \cdots\}$, is evidently transformed into zero by $A^-(s')$,

$$A^-(s')\Phi_0 = 0. \tag{8.2$_0$}$$

This relation is on occasion used to characterize the vacuum state. The one-particle-state $\Phi_1 \leftrightarrow \{0, \psi_1(s_1), 0, \cdots\}$ is transformed by the operator $A^-(s')$ into the state $\psi_1(s')\Phi_0$, which depends on the choice of the value s'. The number $\psi_1(s')$, however, need not be finite for all values of s'; for, the function $\psi_1(s')$ was required merely to be in $\mathfrak{K}_1$, i.e. to be quadratically integrable

$$\int |\psi_1(s_1)|^2 \, dm(s_1) < \infty.$$

The same remark applies, of course, to the representers $\{\psi_{n+1}(s)_n(s')\}$ which we get by applying the operator $A^-(s')$ to the state $\Phi_n \leftrightarrow \{\psi_n(s)_n\}$. For this reason, we must regard $A^-(s')$ as only a symbolic operator.

The symbolic operator $A^-(s')$ can be defined properly in terms of associated linear forms. Let $g(s)$ be any function in $\mathfrak{K}$, i.e. one for which

$$\int |g(s)|^2 \, dm(s) < \infty. \tag{8.3}$$

The state in the space $\mathfrak{G}$ which this function represents will be denoted by g; the state represented by the complex conjugate function $\bar{g}(s)$ will be denoted by $\bar{g}$. Then we define the operator

$$A^-\{\bar{g}\} = \int \bar{g}(s')A^-(s') \, dm(s') \tag{8.4}$$

by the representation

$$\begin{aligned} A^-\{\bar{g}\}\Phi &= \int \bar{g}(s')A^-(s') \, dm(s')\Phi \\ &\leftrightarrow \left\{(n+1)^{1/2} \int \bar{g}(s')\psi_{n+1}(s)_n(s')dm(s')\right\}. \end{aligned} \tag{8.5}$$

Clearly, the function $\int \bar{g}(s')\psi_{n+1}(s)_n(s')dm(s')$ of $(s)_n$ lies in the space $\mathfrak{K}_n$, and therefore the operator $\int \bar{g}(s')A^-(s') \, dm(s')$ is defined in a dense subspace $\mathfrak{H}_1$ of $\mathfrak{H}$. The operator $A^-(s)$ is defined symbolically.

Thus we have met a symbolic operator defined in a strict way. The class of arbitrary functions entering the associated linear form is here well specified;

it is the class $\mathfrak{K}$ of functions $g(s)$ which are quadratically integrable with respect to an appropriate measure function m.

We proceed to determine the adjoint $A^+(s)$ of the operator $A^-(s)$. It is defined through the condition that the operator

$$A^+\{g\} = \int g(s')A^+(s')\, dm(s') \tag{8.6}$$

be the adjoint of the operator $A^-\{\bar{g}\}$, or simply by the condition

$$(\Phi^{(1)}, A^+(s')\Phi) = (A^-(s')\Phi^{(1)}, \Phi) \tag{8.7}$$

with arbitrary $\Phi^{(1)}$, Φ in $\mathfrak{H}$. More properly, $A^+(s)$ is defined by

$$(\Phi^{(1)}, A^+\{g\}\Phi) = (A^-\{\bar{g}\}\Phi^{(1)}, \Phi). \tag{8.8}$$

We shall see that it will make a difference whether or not we restrict the Hilbert space $\mathfrak{H}$ of states Φ by symmetry or anti-symmetry conditions on their representers.

We first introduce the symbolic function $\delta(s' - s)$ by means of the identity

$$\int f(s)\,\delta(s' - s)\, dm(s) = f(s'). \tag{8.9}$$

Note that this δ-function is more general than that of Dirac since it refers to a more general measure function m. If, for example, the spectrum of s is purely discrete, $\delta(s)$ is a "proper" discontinuous function.

Next we infer from relation (8.7) that we can write the expression $(\Phi^{(1)}, A^+(s')\Phi)$ in the form

$$\begin{aligned}(\Phi^{(1)}, A^+(s')\Phi) &= \sum_n (n+1)^{1/2} \int \bar{\psi}^{(1)}_{n+1}(s)_n(s')\psi_n(s)_n\, dm(s)_n \\ &= \sum_n (n+1)^{1/2} \iint \bar{\psi}^{(1)}_{n+1}(s)_{n+1}\delta(s' - s_{n+1})\psi_n(s)_n\, dm(s)_n\, dm(s)_{n+1} \\ &= \sum_n (n+1)^{1/2} \int \bar{\psi}^{(1)}_{n+1}(s)_{n+1}\delta(s' - s_{n+1})\psi_n(s)_n\, dm(s)_{n+1} \\ &= \sum_n n^{1/2} \int \bar{\psi}^{(1)}_n(s)_n\delta(s' - s_n)\psi_{n-1}(s)_{n-1}\, dm(s)_n\, .\end{aligned} \tag{8.10}$$

Here and in the following we should set $\psi_{-1} = 0$. We must make use of the arbitrariness of the function $\psi^{(1)}_n(s)_n$. Note that it does make a difference here whether or not the representers $\psi^{(1)}_n(s)_n$ have symmetry properties.

If we do not make any symmetry restrictions on these representers we may infer from (8.10) the relation

$$A^+(s')\Phi \leftrightarrow \{n^{1/2}\delta(s' - s_n)\psi_{n-1}(s)_{n-1}\}.$$

If the representers are to be symmetric, the state $A^+\Phi$ is represented as

$$(8.11)_{Sy} \qquad A^+(s')\Phi \leftrightarrow \{n^{1/2} Sy\ \delta(s' - s_n)\psi_{n-1}(s)_{n-1}\}.$$

$Sy\ f(s)_n$ denotes the symmetric part of the function $f(s)_n$ obtained by dividing by $n!$ the sum of the functions which result by all permutations of $(s_1, \cdots, s_n)$ from $f(s_1, \cdots, s_n)$. Thus $A^+(s')\Phi$ is represented by symmetric functions.

Similarly, if we wish to use only anti-symmetric functions, we have

$$(8.11)_{Asy} \qquad A^+(s')\Phi \leftrightarrow \{Asy\ n^{1/2}\delta(s' - s_n)\psi_{n-1}(s)_{n-1}\};$$

$Asy\ f(s)_n$ indicates the anti-symmetric part of the function $f(s)_n$ obtained by dividing the alternating sum of all functions resulting from permutation of the variables $s_1, s_2, \cdots, s_n$ by $n!$; the term alternating indicates that these functions are to be taken with the positive sign if the permutation is even and with the negative if it is odd.

Note, in particular, that the stipulation $\psi_{-1} = 0$ expresses the fact that the 0-component of $A^+(s')\Phi$ vanishes for either fermions, bosons, or non-symmetrized particles.

$$(8.11)_0 \qquad (A^+(s')\Phi)_0 = 0.$$

Since the operator $A^+(s)$ transforms a number eigenstate corresponding to n particles into one for $n + 1$ particles, $A^+(s)$ is called a *creation operator*. Of course, $A^+(s)$ is a symbolic operator; in a sense it is even more so than A^-, since its definition involves the introduction of a δ-function. With the aid of a function $g(s)$ in $\mathfrak{R}$ we can form the proper creation operator

$$(8.12) \qquad A^+\{g\} = \int g(s')A^+(s')\, dm(s')$$

with the representation

$$(8.13)_{Sy} \qquad A^+\{g\}\Phi \leftrightarrow \{n^{1/2}\ Sy\ g(s_n)\psi_{n-1}(s)_{n-1}\}$$

for bosons, and

$$(8.13)_{Asy} \qquad A^+\{g\}\Phi \leftrightarrow \{n^{1/2}\ Asy\ g(s_n)\psi_{n-1}(s)_{n-1}\}$$

for fermions.

It could be shown that the operator $A^+\{g\}$ defined by (8.13) is closed in the same subspace $\mathfrak{H}_1$ of the Hilbert space $\mathfrak{H}$ as the operator $A^-\{\bar{g}\}$, and that the two operators $A^+\{g\}$ and $A^-\{\bar{g}\}$ in this subspace $\mathfrak{H}_1$ are adjoint to each other in the strict sense of the theory of operators in Hilbert space. This fact could easily be derived from the identity

$$\| A^-\{\bar{g}\}\Phi \|^2 \mp \| A^+\{g\}\Phi \|^2 = \int | g |^2 \, dm(s),$$

which follows from the commutation relations (8.16) or (8.20) to be derived later on. Incidentally, a similar statement does not hold for operators A^+ defined without symmetry or anti-symmetry conditions.

Of considerable importance are the operators $A^+(s')A^-(s'')$ and $A^-(s'')A^+(s')$ obtained by successive application of an annihilation and a creation operator. Of course, these products have only symbolic meaning and are associated with proper operators which are bilinear forms.

In the case of bosons the representation

$$A^+(s')A^-(s'')\Phi \leftrightarrow \{n^{1/2} \, Sy \, \delta(s' - s_n)\psi_n(s)_{n-1}(s'')\} \tag{8.14}$$

is valid. In order to obtain a representation of the product $A^-(s'')A^+(s')$ we note that by virtue of the symmetry of the function $\psi_n(s)_n$, relation $(8.11)_{sy}$ can be written in the form

$$\begin{aligned} A^+(s')\Phi \leftrightarrow \Big\{\frac{1}{n^{1/2}} \, [\delta(s' - s_n)\psi_{n-1}(s)_{n-1} \\ + (n-1) \; Sy_{n-1} \, \delta(s' - s_{n-1})\psi_{n-1}(s)_{n-2}(s_n)]\Big\}, \end{aligned}$$

in which Sy_{n-1} indicates that the function is to be symmetrized with respect to $s_1, s_2, \cdots, s_{n-1}$, but not s_n. From relation (8.2) we then derive

$$A^-(s'')A^+(s')\Phi \leftrightarrow \{\delta(s' - s'')\psi_n(s)_n + n \; Sy_n \, \delta(s' - s_n)\psi_n(s)_{n-1}(s'')\}. \tag{8.15}$$

Subtracting (8.15) from (8.14) we find

$$[A^-(s'')A^+(s') - A^+(s')A^-(s'')]\Phi \leftrightarrow \{\delta(s' - s'')\psi_n(s)_n\} = \delta(s' - s'')\{\psi_n(s)_n\}$$

or

$$[A^-(s'')A^+(s') - A^+(s')A^-(s'')]\Phi = \delta(s' - s'')\Phi.$$

This relation can be written symbolically as

$$A^-(s'')A^+(s') - A^+(s')A^-(s'') = \delta(s' - s''). \tag{8.16}$$

It is remarkable that the number n does not explicitly occur any more. In fact, the reason for using the factor $(n + 1)^{1/2}$ in the definition of A^- given in

(8.2) was just to achieve this result. In addition to (8.16) we have the obvious relations

$$A^+(s'')A^+(s') - A^+(s')A^+(s'') = 0, \tag{8.16^+}$$

$$A^-(s'')A^-(s') - A^-(s')A^-(s'') = 0. \tag{8.16^-}$$

Relations (8.16) were first explicitly stated by Dirac [8].

Next we discuss fermion fields, whose state representers are anti-symmetric. From (8.2) and $(8.11)_{Asy}$ we immediately derive the relation

$$A^+(s')A^-(s'')\Phi \leftrightarrow \{n^{1/2}\, Asy\; \delta(s' - s_n)\psi_n(s)_{n-1}(s'')\}. \tag{8.17}$$

From the anti-symmetry of $\psi_n(s)_n$ one can also easily derive the following expression for the $(n + 1)$-st component $(A^+(s')\Phi)_{n+1}$ of the state $A^+(s')\Phi$.

$$\begin{aligned}(A^+(s')\Phi)_{n+1} = \frac{1}{(n+1)^{1/2}}\,[\delta(s' - s_{n+1})\psi_n(s)_n \\ - n\; Asy_n\; \delta(s' - s_n)\psi_n(s)_{n-1}(s_{n+1})],\end{aligned} \tag{8.18}$$

in which Asy_n indicates that the function is to be anti-symmetrized with respect to $s_1, s_2, \cdots, s_n$ but not with respect to s_{n+1}. Consequently,

$$A^-(s'')A^+(s')\Phi \leftrightarrow \{\delta(s' - s'')\psi_n(s)_n - n\; Asy\; \delta(s' - s_n)\psi_n(s)_{n-1}(s'')\}. \tag{8.19}$$

From (8.19) and (8.17) one immediately deduces the relation

$$A^-(s'')A^+(s') + A^+(s')A^-(s'') = \delta(s' - s''). \tag{8.20}$$

Thus the commutation relation for fermions differs from that for bosons in the sign of the term $A^+(s')A^-(s'')$. In addition to relation (8.20) relations

$$A^+(s'')A^+(s') + A^+(s')A^+(s'') = 0, \tag{8.20^+}$$

$$A^-(s'')A^-(s') + A^-(s')A^-(s'') = 0 \tag{8.20^-}$$

hold as is easily verified from $(8.11)_{Asy}$ and (8.2). These relations were given by Jordan and Wigner [10].

Let us return to boson fields and define operators $\Xi(s)$ and $\Pi(s)$ which, in the case that s is taken to be x, have the properties required of the operators $\Xi(x)$ and $\Pi(x)$ formulated in Sections 1 to 4. Relation (8.16) is of the same type as the commutation relation (2.1) for the operators $\Xi(x)$ and $\Pi(x)$. We thus see that it is possible to define symbolic operators in a rigorous way so that such commutation rules hold. The only difference so far is that the operators $\Xi(x)$ and $\Pi(x)$ are supposed to be Hermitian while the operators $A^-(s)$ and $A^+(s)$ are not.

Of course from the pair $A^-(s)$, $A^+(s)$ we can construct the pair of operators $B(s)$ and $C(s)$:

$$B(s) = 2^{-1/2}[A^-(s) + A^+(s)], \qquad C(s) = 2^{-1/2}i[A^+(s) - A^-(s)]. \tag{8.21}$$

They are Hermitian in the space $\mathfrak{H}_1$ in the sense that the operators $B\{g\}$ and $C\{g\}$ are Hermitian provided the function $g(s)$ is real. The closures of these operators are easily seen to be spectrally representable.

From the commutation rule (8.16) we immediately derive the relation

$$C(s')B(s'') - B(s'')C(s') = -i\delta(s' - s''), \tag{8.22}$$

which precisely agrees with the rule (2.1) for $s = x$. In fact, in Section 9 we shall give a representation of the operators $\Xi(s)$ and $\Pi(s)$ in terms of $B(s)$ and $C(s)$.

At this place we must add some remarks about the behavior of the operators $A^-(s)$ when a new representation of the states Ψ of $\mathfrak{S}$ is introduced. Let such a state first be represented by functions $\psi(s)$ of a manifold $\mathfrak{K}$, and then by functions $'\psi('s)$ of a manifold $'\mathfrak{K}$ connected with the functions $\psi(s)$ by a linear transformation V,

$$'\psi('s) = V\psi(s). \tag{8.23}$$

The complex conjugate functions are then connected by a complex conjugate transformation,

$$'\overline{\psi}('s) = \overline{V}\overline{\psi}(s). \tag{$\overline{8.23}$}$$

In order that the operator $A^-\{\overline{g}\}$, defined by (8.4), be independent of the representation, it is necessary that the operator $A^-(s)$ be transformed according to

$$'A^-('s) = VA^-(s), \tag{8.24$^-$}$$

while the operator $A^+(s)$ must be transformed according to

$$'A^+('s) = \overline{V}A^+(s). \tag{8.24$^+$}$$

Thus, in order that $A^+\{g\}$ be the adjoint of $A^-\{\overline{g}\}$ independently of the representation the annihilation and creation operators are transformed in the same way as the representers $\psi(s)$ and $\overline{\psi}(s)$ respectively.

It is seen that the operators $B(s)$ and $C(s)$ given by (8.17), are not transformed into the corresponding operators, $'B(s')$, $'C(s')$, when $\psi(s)$ is transformed into $'\psi('s)$, unless the transformation V happens to be real,

$$\overline{V} = V. \tag{8.25}$$

This fact is to be kept in mind later on. If $\overline{V} = V$, the variable $'s$ will be said to be "related in a real manner" to the variable s.

We note, for example, that the position and momentum variables are not related in a real manner to each other. Occasionally, it is desired to employ variables s of which the energy is a function $\omega = \omega(s)$ of a variable s which is related to the position variable x in a real manner. The momentum k can then not be chosen as s, but other such variables can be found. For example, one may take the energy itself and appropriate angular variables. Also when the field is enclosed in a box we can find such variables since there are always real eigenfunctions of the operator Ω^X.

A second important remark concerns the application of operators on the annihilation and creation operators. Let Λ^S be a Hermitian operator acting on functions $\psi(s)$ of the space $\mathfrak{R}'$. Then we define the symbolic operators $\Lambda^S A^{\pm}(s)$ by the identity

$$\int g(s)\Lambda^S A^{\pm}(s)\, ds = \int \Lambda^S g(s)\cdot A^{\pm}(s)\, ds = A^{\pm}\{\Lambda g\}, \tag{8.26}$$

involving an arbitrary function $g(s)$ in the space $\mathfrak{R}'$. We then say that the operators $\Lambda^S A^{\pm}(s)$ transform the state Φ into the states represented by

$$\Lambda^S A^-(s)\Phi \leftrightarrow \{(n+1)^{1/2}\Lambda^S \psi_{n+1}(s)_n(s)\}, \tag{8.27}$$

$$\Lambda^S A^+(s)\Phi \leftrightarrow \left\{n^{1/2}\begin{pmatrix} Sy \\ Asy \end{pmatrix}\Lambda^S \delta(s - s_n)\psi_{n-1}(s)_{n-1}\right\}. \tag{8.28}$$

In the first case we say that Λ^S acts on the function $\psi_{n+1}(s)_n(s)$ only in as much as it is a function of s; in the second case we say that Λ^S acts on the function $\delta(s - s_n)$. Of course, the proper meaning of these expressions is defined by (8.26). We shall employ the definitions (8.27, 8.28) in particular for the operators $\Lambda^S = (\Omega^S)^{1/2}$ and $\Lambda^S = (\Omega^S)^{-1/2}$.

Using an (N,S)-representation of boson states we introduce the operators, see (8.21),

$$\begin{aligned} \Xi(s) &= (\Omega^S)^{-1/2}B(s) = (2\Omega^S)^{-1/2}A^-(s) + (2\Omega^S)^{-1/2}A^+(s), \\ \Pi(s) &= (\Omega^S)^{1/2}C(s) = -i(\Omega^S/2)^{1/2}A^-(s) + i(\Omega^S/2)^{1/2}A^+(s), \end{aligned} \tag{8.29}$$

assuming that the variable s is related in a real manner to the position variable x. Certainly, as seen from (8.18) or (8.16), these operators satisfy the same commutation rule

$$\Pi(s')\Xi(s'') - \Xi(s'')\Pi(s') = -i\delta(s' - s''), \tag{8.30}$$

as the operators $\Pi(x)$, $\Xi(x)$ referred to in Section 2. The reason why the operators (8.29) and not the operators B, C are identified with the operators Π and Ξ of Section 2 will be explained in the next section.

9. *Time Variation of Annihilation and Creation Operators and Representation of Field Operators*

It is easy to determine the change of the operators $A^-(s)$ and $A^+(s)$ in the course of time. We adopt here the Heisenberg picture in which the state does not vary in time. The state may then be represented by functions of the eigenvalues of a complete system of commuting variables S associated with an initial time, $t = 0$.

Let L be an operator acting on the states Φ of the field representing an observable at the time $t = 0$. Let the energy operator associated with the field be $[\Omega]$, defined by the representation (6.35) or (6.37). We assume this energy operator to be constant in time. In Heisenberg's picture, in which operators are supposed to act on the fixed initial state Φ, the change of the operator L in the course of time is given by

$$L(t) = \exp\{it[\Omega]\}L\exp\{-it[\Omega]\}. \tag{9.1}$$

Applying this formula, in particular, to the annihilation operator $L = A^-(s)$ using the representation (6.36) and (8.2), we find

$$\exp\{-it[\Omega]\}\Phi \leftrightarrow \{\exp\{-it\,\Sigma^n\Omega^S\}\psi_n(s)_n\},$$

hence

$$A^-(s)\exp\{-it[\Omega]\}\Phi \leftrightarrow \{(n+1)^{1/2}\exp\{-it\,\Sigma^n\Omega^S - it\Omega^S\}\psi_{n+1}(s)_n(s)\},$$

and thus

$$A^-(s,\,t)\Phi \leftrightarrow \{(n+1)^{1/2}\exp\{it\Sigma^n\Omega^S\}\exp\{-it\Sigma^n\Omega^S - it\Omega^S\}\psi_{n+1}(s)_n(s)\}.$$

Cancelling the first exponential function we find

$$A^-(s,\,t)\Phi \leftrightarrow \{(n+1)^{1/2}\exp\{-it\Omega^S\}\psi_{n+1}(s)_n(s)\}, \tag{9.2}$$

whence, by (8.27),

$$A^-(s,\,t) = \exp\{-it\Omega^S\}A^-(s). \tag{9.3}$$

Correspondingly, we find for the creation operators the relation

$$A^+(s,\,t) = \exp\{it\Omega^S\}A^+(s). \tag{9.4}$$

Thus it is seen that the annihilation operator in the Heisenberg picture changes in the course of time in exactly the same way as the state of a single particle, or its probability amplitude, changes in the Schrödinger picture. This remarkable fact, together with the fact that the operators $A^+(s)$ and $A^-(s)$ are transformed in the same way as the representers $\bar{\psi}(s)$ and $\psi(s)$ respectively, contributed to the misunderstanding which in earlier presentations led to the description of the process of second quantization as consisting in making an operator out of a state or a probability amplitude. It appears, though, that this misunderstanding has been a very helpful guide in setting up the equations of interaction between light and matter, in the form which is at present still accepted.

For a boson field we introduced the operators $\Xi(s)$ and $\Pi(s)$ by relations (8.29) at the end of the previous section. Consequently, we deduce from (9.3), (9.4) the relation

$$\Xi(s, t) = (2\Omega^S)^{-1/2} \exp\{-it\Omega^S\} A^-(s) + (2\Omega^S)^{-1/2} \exp\{it\Omega^S\} A^+(s), \tag{9.5}$$

$$\Pi(s, t) = -i(\Omega^S/2)^{1/2} \exp\{-it\Omega^S\} A^-(s) + i(\Omega^S/2)^{1/2} \exp\{it\Omega^S\} A^+(s).$$

In an obvious manner these operators can be differentiated with respect to the time t. The results are

$$\nabla_t \Xi(s, t) = \Pi(s, t), \tag{9.6}$$

$$\nabla_t \Pi(s, t) = -(\Omega^S)^2 \Xi(s, t). \tag{9.7}$$

Relation (9.6) for $t = 0$ agrees with relation (3.3) when we set $s = x$; in order to achieve this agreement it was necessary to adopt the definitions (8.29) of the operators $\Xi(s)$ and $\Pi(s)$. Since $(\Omega^X)^2 = \mu^2 - \nabla^2$ by virtue of assumption (6.13), relations (9.6) and (9.7) combined lead for $s = x$ to the relation

$$(\nabla_t^2 - \nabla^2 + \mu^2)\Xi(x, t) = 0, \tag{9.8}$$

which agrees with the differential equation (3.1). Thus the definitions (8.29) are justified.

In other words, it has been shown by explicit representation that there are field operators Ξ and Π which have the properties described in Section 3.

For fermion fields it is also possible to introduce field quantities satisfying simple commutation rules. It is customary, however, and apparently with good reason, to use the operators $A^-(s)$ and $A^+(s)$ themselves as the basic operators for describing the properties of fermion fields. It is further customary—unfortunately—to denote these operators by $\psi(s)$ and $\psi^*(s)$ respectively, although the interpretation of these operators as having resulted from a second quantization of probability amplitudes has been abandoned.

10. *Trace Operators*

The time dependence of the operators $\Xi(s, t)$ and $\Pi(s, t)$ is given by relations (9.5) with the aid of the unitary operator $\exp\{it[\Omega]\}$. Thus the operator $[\Omega]$ plays the role of the operator J introduced in Section 4. In this section we shall show that the operator $[\Omega]$ equals the specific operator H defined by (4.15).

The operator $[\Omega]$ was derived from the operator Ω through the process of biquantization described at the end of Section 7. As pointed out by Thomas [14], it is possible to describe the process of second quantization of an operator with the aid of the annihilation and creation operators $A^-(s)$, $A^+(s)$ without specific reference to the configuration space representation of the state Φ. (In special cases this process was already employed by Fock.)

This is done with the aid of the notion of the "trace of a dyadic". As a dyadic we want to understand here a symbolic function $j(s', s'')$ of two variables s', s'' associated with a bounded form, linear in $\bar{g}^{(2)}$ and in $g^{(1)}$, which is written as

$$\iint g^{(1)}(s')j(s', s'')\bar{g}^{(2)}(s'')\, dm(s')\, dm(s'').$$

For example, the symbolic function $\delta(s' - s'')$ is such a dyadic associated with the bilinear form

$$(10.1) \qquad \int g^{(1)}(s)\bar{g}^{(2)}(s)\, dm(s) = \iint \delta(s' - s'')g^{(1)}(s')\bar{g}^{(2)}(s'')\, dm(s')\, dm(s'').$$

For certain such dyadics we can form the "trace" Trj, symbolically written as

$$(10.2) \qquad Trj = \int j(s, s)\, dm(s),$$

in the following manner. Let $u_\sigma(s)$, $\sigma = 1, 2, \cdots$, be a complete orthonormal system of quadratically integrable functions in $\mathfrak{R}$. Then we set

$$(10.3) \qquad j_{\sigma'\sigma''} = \int u_{\sigma'}(s')j(s', s'')\bar{u}_{\sigma''}(s'')\, dm(s')\, dm(s'')$$

and define

$$(10.4) \qquad Trj = \sum_{\sigma=1}^{\infty} j_{\sigma\sigma},$$

provided that the series here converges absolutely. It can be shown that the value of Trj is independent of the choice of the orthonormal system; see e.g. [19].

Taking as measure function the set function μ which is equal to the number of positive integers in the set, we may write relation (10.4) in the form

$$Trj = \int j_{\sigma\sigma}\, d\mu(\sigma);$$

the right member here is a special form of the right member of formula (10.2). Since the trace is independent of the choice of the coordinate system we may adopt just as well the notation (10.2), which refers to a representation in terms of an arbitrary set of operators S.

If the function $j(s, s)$ is integrable with respect to the measure function $m(s)$, the trace Trj is given by relation (10.2) when the right member of it is understood in a proper sense. We shall not prove this here, and also leave open whether or not the function $j(s, s)$ is integrable. For the sake of simplicity we proceed in the following arguments as if this were the case; it is not difficult to make the arguments rigorous by going back to the definition (10.4) of the trace.

That not every dyadic possesses a trace is exemplified by the dyadic $j(s', s'') = \delta(s' - s'')$. The components $j_{\sigma\sigma}$ with respect to an orthonormal system are evidently

$$j_{\sigma\sigma} = \iint \delta(s' - s'')u_\sigma(s')\bar{u}_\sigma(s'')\, dm(s')\, dm(s'') = \int u_\sigma(s)\bar{u}_\sigma(s)\, dm(s) = 1$$

and hence the trace is $\sum_1^\infty j_{\sigma\sigma} = \infty$. On the other hand the dyadic formed by every product $\bar{f}^{(1)}(s')f^{(2)}(s'')$ of two functions in $\mathfrak{K}$ evidently possesses a trace; the same is true for a finite sum and for certain infinite series of such products.

We shall now show how the trace operator may be used to describe the biquantization of operators.

To this end we consider a spectrally representable operator Λ acting on the states of a subspace $\mathfrak{S}'$ of the space $\mathfrak{S}$. For the following it is convenient to consider the operator Λ as the product of two Hermitian operators acting in appropriate subspaces

$$\Lambda = \Lambda_1\Lambda_2 \,. \tag{10.5}$$

On occasion we shall set $\Lambda_1 = \Lambda$, $\Lambda_2 = 1$ or $\Lambda_1 = 1$, $\Lambda_2 = \Lambda$. In the following we write Λ^s instead of Λ^S, for, then we can distinguish between $\Lambda^{s'}g(s', s'')$ and $\Lambda^{s''}g(s', s'')$: the operator $\Lambda^{s'}$ acts on $g(s', s'')$ inasmuch as it is a function of s', while $\Lambda^{s''}$ acts on $g(s', s'')$ as a function of s''.

We now form the operators $\Lambda_2^{s''}A^-(s'')$ and $\Lambda_1^{s'}A^+(s')$, see (8.27), and the dyadic

$$(\Phi^{(1)}, \Lambda_1^{s'}A^+(s')\Lambda_2^{s''}A^-(s'')\Phi^{(2)})$$

with the aid of two states $\Phi^{(1)}$ and $\Phi^{(2)}$. In order to evaluate the trace

$$\int (\Phi^{(1)}, \Lambda_1^{s}A^+(s)\Lambda_2^{s}A^-(s)\Phi^{(2)})\, dm(s)$$

of this dyadic one need only verify that it can be written in the form

$$
\begin{aligned}
&(\Phi^{(1)}, \Lambda_1^{s'} A^+(s')\Lambda_2^{s''} A^-(s'')\Phi^{(2)}) \\
&\quad = \sum_n n \int \overline{\psi_n^{(1)}(s)_n}\, \Lambda_1^{s'} \, \begin{matrix} Sy \\ Asy \end{matrix} \, \delta(s' - s_n)\Lambda^{s''} \psi_n(s)_{n-1}^{(2)}(s'')\, dm(s)_n \\
&\quad = \sum_n n \int \Lambda_1^{s'}\, \overline{\psi_n^{(1)}(s)_{n-1}(s')}\, \Lambda_2^{s''} \psi_n^{(2)}(s)_{n-1}(s'')\, dm(s)_{n-1} \, .
\end{aligned}
$$

In the second line Sy and Asy refer to bosons and fermions respectively; in the third line the symmetric or antisymmetric character of the function ψ_n is used. One now uses the definition (10.4) of the trace and verifies that the order of the summations may be interchanged. The trace of the dyadic is found to be

$$
\sum_n n \int \overline{\psi}_n^{(1)}(s)_n \Lambda_1^{s_n} \Lambda_2^{s_n} \psi^{(2)}(s)_n \, dm(s)_n = \sum_n n \int \overline{\psi}_n^{(1)}(s)_n \Lambda^{s_n} \psi^{(2)}(s)_n \, dm(s)_n \, ,
$$

or, what is the same because of the symmetry or anti-symmetry,

$$
\int (\Phi^{(1)}, \Lambda_1^{s} A^+(s)\Lambda_2^{s} A^-(s)\Phi^{(2)})\, dm(s) = \sum\nolimits_n \int \psi_n^{(1)}(s)_n (\Sigma^n \Lambda^s)\psi^{(2)}(s)_n \, dm(s)_n \, .
$$

By definition (6.35) of the operator $[\Lambda]$ resulting from biquantization of Λ, the last relation can be written as

$$
\begin{aligned}
&Tr(\Phi^{(1)}, \Lambda_1^{s'} A^+(s')\Lambda_2^{s''} A^-(s'')\Phi^{(2)}) \\
&\quad = \int (\Phi^{(1)}, \Lambda_1^{s} A^+(s)\Lambda_2^{s} A^-(s)\Phi^{(2)})\, dm(s) \qquad (10.6) \\
&\quad = (\Phi^{(1)}, [\Lambda]\Phi^{(2)}),
\end{aligned}
$$

or, symbolically, as

$$
Tr \Lambda_1^{s'} A^+(s')\Lambda_2^{s''} A^-(s'') = \int \Lambda_1^{s} A^+(s)\Lambda_2^{s} A^-(s)\, dm(s) = [\Lambda_1\Lambda_2] = [\Lambda]. \tag{10.7}
$$

This formula shows that for bosons and fermions the process of second quantization of an operator $\Lambda = \Lambda_1\Lambda_2$ can be described with the aid of the annihilation and creation operators without reference to the particle representation. At the same time it is seen to be irrelevant how the operator Λ is split up as the product of two operators acting on $A^+(s)$ or on $A^-(s)$ respectively.

Specifying the operator $\Lambda_1\Lambda_2$ to be the identity 1 or the energy operator Ω we find, in particular,

$$
\int A^+(s)A^-(s)\, dm(s) = [1] = N, \tag{10.8}
$$

$$(10.9)\qquad \int A^+(s)\Omega^S A^-(s)\,dm(s) = \int (\Omega^S)^{1/2}A^+(s)(\Omega^S)^{1/2}A^-(s)\,dm(s) = [\Omega].$$

One may naturally ask why one has not considered the "inverted" trace

$$\int \Lambda_1^S A^-(s)\Lambda_2^S A^+(s)\,dm(s)$$

instead of the trace (10.7). It is immediately inferred from (8.16) or (8.20) and (10.7) that this trace would be equal to

$$\delta(0)\int \Lambda^S\,dm(s) \pm [\Lambda_1\Lambda_2];$$

the first term here is infinite unless both the measure function m has a jump at $s = 0$, so that $\delta(0)$ is finite, and $\int \Lambda^S\,dm(s)$ is finite. Since this will in general not be the case the "inverted" trace will generally be infinite.

In the case of bosons, it is of particular interest to express the traces $[1] = N$ and $[\Omega]$ in terms of the operators $\Xi(s)$ and $\Pi(s)$ defined by (8.29). From this relation we first derive

$$(10.10)\qquad \begin{aligned} A^-(s) &= (\Omega^S/2)^{1/2}\Xi(s) + i(2\Omega^S)^{-1/2}\Pi(s),\\ A^+(s) &= (\Omega^S/2)^{1/2}\Xi(s) - i(2\Omega^S)^{-1/2}\Pi(s). \end{aligned}$$

Insertion into (10.8) and (10.9) gives

$$(10.11)\qquad \begin{aligned} N &= \frac{1}{2}\int [(\Omega^S)^{1/2}\Xi(s) - i(\Omega^S)^{-1/2}\Pi(s)][(\Omega^S)^{1/2}\Xi(s) + i(\Omega^S)^{-1/2}\Pi(s)]\,dm(s)\\ &= \frac{1}{2}\int (\Omega^S)^{-1}[\Omega^S\Xi(s) - i\Pi(s)][\Omega^S\Xi(s) + i\Pi(s)]\,dm(s) \end{aligned}$$

and

$$(10.12)\qquad [\Omega] = \frac{1}{2}\int [\Omega^S\Xi(s) - i\Pi(s)][\Omega^S\Xi(s) + i\Pi(s)]\,dm(s).$$

Here we have made use of the fact that it is irrelevant whether the operator $(\Omega^S)^{1/2}$ acts on the first or on the second factor. Setting $\Omega^X = (\mu^2 - \nabla^2)^{1/2}$, and $\Pi = \nabla_t\Xi$, the expression (10.12) agrees with the expression (4.15) for the modified energy operator H. Thus we have shown that the expression H for the intrinsic energy operator as given in (4.15) is the correct biquantized energy operator; it is to be taken as operator J instead of the energy integral $\hat{H}$ when time variations are to be expressed with the aid of a unitary operator $\exp\{itJ\}$.

11. *Oscillators*

Frequently, the field is described as consisting of a manifold of oscillators see Dirac [2]. An oscillator with the mass m is a particle, characterized by a displacement operator Q, a momentum operator P and an energy operator

$$\frac{1}{2}\left(\frac{1}{m}P^2 + m\omega^2Q^2\right)$$

where ω is a given constant. The momentum and displacement operators are connected by the commutation rule

$$PQ - QP = -i.$$

The energy operator then has a discrete spectrum with the eigenvalues $(\nu + 1/2)\omega$, $\nu = 0, 1, 2, \cdots$. In a Q-representation, i.e. a representation of the states by functions $\psi(q)$, the normed eigenfunctions are Hermite functions

$$\psi_\nu(q) = H_\nu[(m\omega)^{1/2}q] \exp\left\{-\frac{1}{2}m\omega q^2\right\}, \tag{11.1}$$

involving the Hermite polynomials H_ν.

Instead of the energy operators given, one may consider the energy operator

$$\frac{1}{2}\left(\frac{1}{m}P^2 + m\omega^2Q^2\right) - \frac{1}{2}\omega,$$

differing from the original one by a constant so that the eigenvalues of the new energy operator are just $\nu\omega$, $\nu = 0, 1, \cdots$. By virtue of the commutation rules, the new energy operator can also be written in the form

$$\frac{m}{2}\left(\omega Q - \frac{i}{m}P\right)\left(\omega Q + \frac{i}{m}P\right).$$

We can see now in which sense the field can be described as an assembly of oscillators. Suppose we choose a spectral representation of the energy operator Ω, i.e. we choose the operators S such that Ω is a function $\omega(S)$ of S and that at the same time S is related to the position X in a real manner in a sense introduced by (8.25). According to relation (10.12), the field energy operator $[\Omega]$ then assumes the form

$$[\Omega] = \frac{1}{2}\int [\omega(s)\Xi(s) - i\Pi(s)][\omega(s)\Xi(s) + i\Pi(s)]\,dm(s). \tag{11.2}$$

To every value of s we now may assign an "oscillator" with the displacement $\Xi(s)$, the momentum density $\Pi(s)$, and the energy density

$$\tfrac{1}{2}[\omega(s)\Xi(s) - i\Pi(s)][\omega(s)\Xi(s) + i\Pi(s)].$$

The commutation rule (7.7) for $s' \neq s''$ expresses the independence of two different oscillators, while it corresponds to the commutation rule for P and Q for $s' = s'' = s$.

If the spectrum of the energy operator Ω and of the parameter operator S is discrete, which will be the case if the field is enclosed in a box, the set of oscillators is discrete. In that case an eigenstate of the field may be described by saying that each oscillator associated with a value of s is in a state characterized by the "level" $\nu = \nu(s)$ and has the energy $\nu(s)\omega(s)$. Incidentally, this level $\nu(s)$ is just the occupation number in a particle description, i.e. the number of particles which are in the eigenstate of the operator S with the eigenvalue s; see Section 6.

In case the spectrum of the energy operator Ω is not discrete, such a description is difficult since then the level $\nu(s)$ might have to change discontinuously, if $\omega(s)$ changes continuously. Nevertheless, this is possible as will be shown in Part IV.

Another obstacle in case the energy Ω does not have a discrete spectrum would seem to occur if one tries to employ a Ξ-representation, i.e. a representation of the states Φ of the field in terms of functions of all the oscillator displacements $\Xi(s)$. This would in fact be the representation which was referred to in the first section. For a single oscillator, the transition from an energy representation to a displacement representation is effected with the aid of the Hermite functions (11.1) depending on the level ν. There would seem to be difficulties in case the level $\nu = \nu(s)$ depended on a continuous parameter s. Nevertheless, as will be shown in the next section, such a Ξ-representation can be given quite independently of the nature of the spectrum of S by using an approach to the description of Hermite polynomials in which reference to the level is omitted. A description in terms of oscillators, however, is then no longer appropriate.

12. *Hermite Functionals and Integration over the Hilbert Space*

The aim of this section is to establish a representation of states by functionals $\phi\{\xi\}$ which yields the simultaneous representation of the field operators $\Xi(x)$. In other words, the aim is to answer the question posed in Section 1. While we shall achieve this aim, in a certain sense at least, we shall realize, at the same time, that such a representation is not at all a suitable tool for the treatment of field quantum theory. In fact, in the discussion of the subsequent parts of this exposition we shall not at all employ this representation. We shall employ the particle representation instead.

Let us consider the space $\mathfrak{K}_n$, introduced previously, which consists of functions of n variables $f(s_1, s_2, \cdots, s_n) = f(s)_n$ which are quadratically integrable with respect to a measure function, i.e.

$$\begin{aligned}(12.1)\qquad &\int\cdots\int |f(s_1, s_2, \cdots, s_n)|^2 \, dm(s_1)\cdots dm(s_n)\\ &= \int |f(s)_n|^2 \, dm(s)_n < \infty .\end{aligned}$$

In particular we shall denote by $\eta(s)$ *real* functions in the space $\mathfrak{R}$ and by $e_n(s_1, s_2, \cdots, s_n) = e_n(s)_n$ *symmetric* functions in the space $\mathfrak{R}_n$. The functions $e_n(s)_n$ are called *n-ics*. For $n = 0$ the symbol $e_0(s)_0$ means just a complex number e_0.

With the aid of an *n-ic* $e_n(s)_n$ we can set up the form of n-th degree $e_n\{\eta\}$ defined by

$$\begin{aligned}(12.2)\qquad e_n\{\eta\} &= \int\cdots\int e_n(s_1, \cdots, s_n)\eta(s_1)\cdots\eta(s_n)\, dm(s_1)\cdots dm(s_n)\\ &= \int e_n(s)_n \eta^n(s)_n \, dm(s)_n , \qquad n > 0\end{aligned}$$

and

$$e_0\{\eta\} = e_0 .$$

In an obvious way we extend the notion of a functional derivative defined by (2.2) so that it refers to integration with respect to a measure function $m(s)$. We see that the functional derivative with respect to η of the form $e_n\{\eta\}$ is

$$\begin{aligned}\frac{\delta}{\delta\eta(s')} e_n\{\eta\} &= \frac{\delta e_n\{\eta\}}{\delta\eta(s')}\\ (12.3)\qquad &= n\int\cdots\int e_n(s_1, \cdots, s_{n-1}, s')\eta(s_1)\cdots\eta(s_{n-1})\, dm(s_1)\cdots dm(s_{n-1})\\ &= n\int e_n(s)_{n-1}(s')\eta^{n-1}(s)_{n-1}\, dm(s)_{n-1} .\end{aligned}$$

The derivative is evidently a function of s' belonging to the space $\mathfrak{R}$.

The operation of taking the derivative of an n-ic form may be considered as a method of constructing an $(n - 1)$-ic form from an n-ic form. The resulting $(n - 1)$-ic form, however, is to be understood only in a symbolic sense; i.e. only the form

$$(12.4)\qquad \int \frac{\delta e_n\{\eta\}}{\delta\eta(s')}\gamma(s')\, dm(s') = n\iint e_n(s)_{n-1}(s')\eta^{n-1}(s)_{n-1}\gamma(s')\, dm(s)_{n-1}\, dm(s')$$

has a proper meaning for functions $\gamma(s)$ in $\mathfrak{R}$. For, the $(n - 1)$-ic $\delta e_n\{\eta\}/\delta\eta(s')$ need not be a proper function of s'.

Next we consider an operation of constructing an $(n + 1)$-ic form from an n-ic form, viz. simply the multiplication of $e_n\{\eta\}$ by $\eta(s')$. That is, $\eta(s')e_n\{\eta\}$ is an $(n + 1)$-ic form obtained from the n-ic form $e_n\{\eta\}$. As in the case of the $(n - 1)$-ic form obtained by taking the derivative, the $(n + 1)$-ic form $\eta(s')e_n\{\eta\}$ is to be understood only in the symbolic sense so that only the expression

$$\int \gamma(s')\eta(s')e_n\{\eta\}\, dm(s') = \int \gamma(s')\eta(s')\, dm(s')e_n\{\eta\} \tag{12.5}$$

has meaning.

Suppose a state Φ of a field is represented by a set of symmetric functions $\psi_n(s)_n = e_n(s)_n$, in $\mathfrak{R}_n$, $n = 0, 1, \cdots$, with e_0 being a constant. Then we may just as well consider the set of forms $e_n\{\eta\}$ with $e_0\{\eta\} = e_0$ as the representer of the state Φ. The $(n + 1)$-ic form (12.5), for example, is derived in this way from the symmetric function $Sy\; e_n(s_1, \cdots, s_n)\gamma(s_{n+1})$. The effect of applying annihilation or creation operators on the state Φ can then easily be described in terms of operations of these n-ic forms $e_n\{\eta\}$. From relation (8.2) and $(8.11)_{Sy}$ we deduce that the representations

$$\begin{aligned} A^-(s')\Phi &\leftrightarrow \left\{\frac{1}{(n + 1)^{1/2}}\,\frac{\delta}{\delta\eta(s')}\, e_{n+1}\{\eta\}\right\}, \\ A^+(s')\Phi &\leftrightarrow \{n^{1/2}\eta(s')e_{n-1}\{\eta\}\} \end{aligned} \tag{12.6}$$

obtain, when Φ is represented as $\Phi \leftrightarrow \{e_n\{\eta\}\}$.

The representation of the state Φ through these functionals $e_n\{\eta\}$ is equivalent to, but different from, the representation of Φ through functions $\psi_n(s)_n$.

Instead of representing the state Φ by a set of functionals, such as $e_n\{\eta\}$, we can represent Φ by a single functional expressed in terms of a series of functionals $e_n\{\eta\}$. In order to develop such a representation we shall introduce Hermite functionals.

First we consider the second derivative

$$\begin{aligned} \frac{\delta^2}{\delta(\eta'')\,\delta(\eta')}\, e_n\{\eta\} &= \frac{\delta^2 e_n\{\eta\}}{\delta(\eta'')\,\delta(\eta')} \\ &= n(n - 1) \int \cdots \int e_n(s_1, \cdots, s_{n-2}, s'', s')\eta(s_1) \cdots \eta(s_{n-2})\, dm(s_1) \cdots dm(s_{n-2}) \\ &= n(n - 1) \int e_n(s)_{n-2}(s'', s')\eta^{n-2}(s)_{n-2}\, dm(s)_{n-2} \end{aligned} \tag{12.7}$$

of a form $e_n\{\eta\}$; it is a function in $\mathfrak{R}_2$ and can be considered a dyadic. We assume that the trace

$$Tr\, \frac{\delta^2}{\delta\eta^2}\, e_n\{\eta\} = \int \frac{\delta^2}{\delta\eta(s)^2}\, e_n\{\eta\}\, dm(s) \tag{12.8}$$

of this dyadic exists. It represents a form of degree $n - 2$. The corresponding $(n - 2)$-ic function may be written symbolically in the form

$$n(n-1)\int e_n(s_1, \cdots, s_{n-2}, s, s)\, dm(s).$$

We further assume that the second derivative of this $(n - 2)$-ic form in its turn possesses a trace which is a form of degree $n - 4$ written as

$$\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^2 e_n\{\eta\} = \iint \frac{\delta^2}{\delta\eta(s')^2}\frac{\delta^2}{\delta\eta(s'')^2} e_n\{\eta\}\, dm(s')\, dm(s''). \tag{12.9}$$

Moreover, we assume that all traces $(Tr\, \delta^2/\delta\eta^2)^\rho\, e_n\{\eta\}$ exist as forms of degree $n - 2\rho$. Clearly,

$$\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^\rho e_n\{\eta\} = 0 \qquad \text{when} \qquad 2\rho > n.$$

We define as a general "Hermite polynomial" of degree n, assigned to the n-ic form $e_n\{\eta\}$, the expression

$$\begin{aligned} He_n\{\eta\} &= e_n\{\eta\} - \frac{1}{2}\, Tr \frac{\delta^2}{\delta\eta^2} e_n\{\eta\} \\ &\quad + \frac{1}{2\cdot 4}\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^2 e_n\{\eta\} - \frac{1}{2\cdot 4\cdot 6}\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^3 e_n\{\eta\} + \cdots . \end{aligned} \tag{12.10}$$

Before justifying this definition we formulate a few important properties of the general Hermite polynomials.

We state that the Hermitian polynomials associated with the two forms $\delta e_n\{\eta\}/\delta\eta(s')$ and $\eta(s')e_n\{\eta\}$ are given by

$$H \frac{\delta}{\delta\eta(s')} e_n\{\eta\} = \frac{\delta}{\delta\eta(s')} He_n\{\eta\}, \tag{12.11}$$

and

$$H\eta(s')e_n\{\eta\} = \left(\eta(s') - \frac{\delta}{\delta\eta(s')}\right) He_n\{\eta\}. \tag{12.12}$$

The first relation is immediately verified from the definition (12.10) by using the identity

$$\frac{\delta}{\delta\eta(s')}\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^\rho e_n\{\eta\} = \left(Tr \frac{\delta^2}{\delta\eta^2}\right)^\rho \frac{\delta}{\delta\eta(s')} e_n\{\eta\}.$$

In order to verify relation (12.11) one need only make use of the identity

$$\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^\rho \eta(s')e_n\{\eta\} = \eta(s')\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^\rho e_n\{\eta\} + 2\rho \frac{\delta}{\delta\eta(s')}\left(Tr \frac{\delta^2}{\delta\eta^2}\right)^{\rho-1} e_n\{\eta\},$$

which is easily established by induction.

Combining relations (12.11) and (12.12), we find

$$H\eta(s')\frac{\delta}{\delta\eta(s'')}e_n\{\eta\} = \left(\eta(s') - \frac{\delta}{\delta\eta(s')}\right)\frac{\delta}{\delta\eta(s'')}He_n\{\eta\}. \tag{12.13}$$

From (12.3) it is evident that

$$Tr\eta(s')\frac{\delta}{\delta\eta(s'')}e_n\{\eta\} = \int \eta(s)\frac{\delta}{\delta\eta(s)}\,dm(s)e_n\{\eta\} = ne_n\{\eta\}, \tag{12.14}$$

whence from (12.13) the relation

$$\int \left(\eta(s) - \frac{\delta}{\delta\eta(s)}\right)\frac{\delta}{\delta\eta(s)}\,dm(s)He_n\{\eta\} = nHe_n\{\eta\} \tag{12.15}$$

follows. It is the basic differential equation satisfied by Hermite polynomials.

In order to justify our definition (12.10) of Hermite polynomials we consider the case that the space $\mathfrak{R}$ is finite dimensional so that the function $\eta(s)$ can be given by a finite number of variables. Specifically, we may assume that s represents one real variable which runs only over a finite number of integers $s = 1, \cdots, r$. That means the measure function $m(\mathfrak{S})$ equals the number of these integers in the set. Instead of $\eta(s)$ we then write η_s. The inner product becomes

$$\int \eta(s)\eta^{(1)}(s)\,dm(s) = \sum_{s=1}^{r} \eta_s\eta_s^{(1)}. \tag{12.16}$$

The Hermite polynomial $He_n\{\eta\}$ becomes a function of r variables

$$He_n\{\eta\} = h_n(\eta_1, \cdots, \eta_r).$$

The functional derivative becomes the partial derivative

$$\frac{\delta}{\delta\eta(s)}He_n\{\eta\} = \frac{\partial}{\partial\eta_s}h_n(\eta_1, \cdots, \eta_r)$$

and differential equation (12.15) goes over into

$$\sum_s \left(\eta_s - \frac{\partial}{\partial\eta_s}\right)\frac{\partial}{\partial\eta_s}h_n = nh_n$$

or

$$\sum_s \left(\eta_s\frac{\partial}{\partial\eta_s} - \frac{\partial^2}{\partial\eta_s^2}\right)h_n = nh_n. \tag{12.17}$$

We maintain that this is exactly the differential equation satisfied by the linear combinations of basic Hermite polynomials.

A basic Hermite polynomial[5] of r independent variables $\eta_1, \cdots, \eta_r$ is a product

$$h(\eta_1, \cdots, \eta_r) = H_{\nu_1}(\eta_1) \cdots H_{\nu_r}(\eta_r)$$

of Hermite polynomials $H_\nu(\eta)$ of degree $\nu_1, \cdots, \nu_r$. Its total degree is $n = \nu_1 + \cdots + \nu_r$.

Since the individual Hermite polynomial $h(\eta) = H_\nu(\eta)$ satisfies the differential equations

$$\left(\eta \frac{d}{d\eta} - \frac{d^2}{d\eta^2}\right)h(\eta) = \nu h(\eta),$$

the product $h(\eta_1, \cdots, \eta_r)$ of such Hermite polynomials satisfies the differential equation (12.17). The same is therefore also true for each linear combination of basic polynomials of total degree n.

Conversely, it can be shown that every polynomial solution of differential equation (12.17) is such a linear combination.

It should be mentioned that in our definition (12.13) of Hermite polynomials the function $e_n(s)_n$ takes the place of the coefficients in such a linear combination of basic polynomials.

We now return to our general Hermite polynomials and form with their aid the "Hermite functionals"

$$Je_n\{\eta\} = (n!)^{-1/2} He_n\{\eta\} \exp\{-(\eta, \eta)/4\} \tag{12.18}$$

with

$$(\eta, \eta) = \int \eta^2(s)\, dm(s). \tag{12.19}$$

The functional derivative of (η, η) is obviously

$$\frac{\delta}{\delta\eta(s)}(\eta, \eta) = 2\eta(s). \tag{12.20}$$

According to the definition (2.2) of the variation $\delta/\delta\eta(s)$ we have

$$\begin{aligned} \frac{\delta}{\delta\eta(s)} \exp\{-(\eta, \eta)/4\} &= -\frac{1}{4} \exp\{-(\eta, \eta)/4\} \frac{\delta}{\delta\eta}(\eta, \eta)\} \\ &= -\frac{1}{2}\eta(s) \exp\{-(\eta, \eta)/4\}. \end{aligned} \tag{12.21}$$

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[5] An elegant treatment of basic Hermite polynomials was given by Grad [23]. Basic polynomials were also employed by Cameron and Martin [22] in a theory of Hermite polynomials of infinitely many variables.

From the definition of the Hermite functionals (12.18) and from the fact that $\delta e_n\{\eta\}/\delta\eta(s)$ is a form of order $(n - 1)$, see (12.3), we derive the relation

$$J\frac{\delta}{\delta\eta(s)}e_n\{\eta\} = [(n-1)!]^{-1/2}H\frac{\delta}{\delta\eta(s)}e_n\{\eta\}\exp\{-(\eta,\eta)/4\}.$$

Furthermore, since $\eta(s)e_n\{\eta\}$ is a form of order $n + 1$, we have

$$J\eta(s)e_n\{\eta\} = [(n+1)!]^{-1/2}H\eta(s)e_n\{\eta\}\exp\{-(\eta,\eta)/4\}.$$

Using the properties of differentiation and of the Hermite functionals as given above we see that (12.11) and (12.12) go over into

$$n^{-1/2}J\frac{\delta}{\delta\eta(s')}e_n\{\eta\} = \left(\frac{1}{2}\eta(s') + \frac{\delta}{\delta\eta(s')}\right)Je_n\{\eta\} \tag{12.22}$$

$$(n+1)^{1/2}J\eta(s')e_n\{\eta\} = \left(\frac{1}{2}\eta(s') - \frac{\delta}{\delta\eta(s')}\right)Je_n\{\eta\}. \tag{12.23}$$

From these relations and from (12.15) it is seen that $Je_n\{\eta\}$ satisfies the differential equation

$$\int\left(\frac{1}{2}\eta(s) - \frac{\delta}{\delta\eta(s)}\right)\left(\frac{1}{2}\eta(s) + \frac{\delta}{\delta\eta(s)}\right)dm(s)Je_n\{\eta\} = nJe_n\{\eta\}. \tag{12.24}$$

This equation is the analogue of the self-adjoint differential equation

$$\left(\frac{1}{2}\eta - \frac{d}{d\eta}\right)\left(\frac{1}{2}\eta + \frac{d}{d\eta}\right)H_\nu(\eta)\exp\{-\eta^2/4\} = \nu H_\nu(\eta)\exp\{-\eta^2/4\},$$

satisfied by the standard Hermitian functions of a single real variable η.

An important property of the Hermite functions $H_\nu(\eta)\exp\{-\eta^2/4\}$ of one variable η is that they are orthogonal to each other:

$$\int H_\nu(\eta)H_\mu(\eta)\exp\{-\eta^2/2\}\,d\eta = 0, \qquad \mu \neq \nu,$$

$$\int H_\nu^2(\eta)\exp\{-\eta^2/2\}\,d\eta = \nu!(2\pi)^{1/2}.$$

It is not difficult to derive from these relations the corresponding relations for general Hermite functionals, provided that the function $\eta(s)$ is given by a finite number of values $\eta(s) = \eta_s$, $s = 1, \cdots, r$, such that

$$(\eta,\eta) = \sum_{s=1}^{r}\eta_s^2 = \eta_1^2 + \eta_2^2 + \cdots + \eta_r^2.$$

Using the abbreviation

$$d\eta = d\eta_1 \cdots d\eta_r$$

and

$$\theta = (2\pi)^{-1/2}, \tag{12.25}$$

the desired relations are

$$\begin{aligned}
\theta^r \iint \cdots \int J\bar{e}_n^{(1)}\{\eta\} Je_m^{(2)}\{\eta\}\, d\eta_1\, d\eta_2 \cdots d\eta_r \\
&= \int J\bar{e}_n^{(1)}\{\eta\} Je_m^{(2)}\{\eta\}\, d(\theta\eta) \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad m \neq n, \\
&= \sum_{s_1, \cdots, s_n = 1}^{r} \bar{e}^{(1)}(s_1, \cdots, s_n) e^{(2)}(s_1, \cdots, s_n) \quad \text{if} \quad m = n.
\end{aligned}$$

The last sum is evidently a special case of the integral

$$\int \bar{e}^{(1)}(s)_n e^{(2)}(s)_n\, dm(s)_n .$$

This fact motivates the definition

$$\begin{aligned}
(n!m!)^{-1/2} \int H\bar{e}_n^{(1)}\{\eta\} He_m^{(2)}\{\eta\} \exp\{-(\eta, \eta)/2\}\, d(\theta\eta) \\
&\equiv \int J\bar{e}_n^{(1)}\{\eta\} Je_m^{(2)}\{\eta\}\, d(\theta\eta) \\
&= 0 \qquad\qquad\qquad\qquad \text{if} \quad m \neq n, \\
&= \int \bar{e}_n^{(1)}(s)_n\, e_n^{(2)}(s)_n\, dm(s)_n \quad \text{if} \quad m = n.
\end{aligned} \tag{12.26}$$

Next let $\{e_n(s)_n\}$ be a sequence of n-ic functions for which

$$\sum_{n=0}^{\infty} \int |\, e_n(s)_n \,|^2\, dm(s)_n < \infty . \tag{12.27}$$

Suppose that for all these n-ics the traces $(Tr\ \delta/\delta\eta^2)^\nu$, $\nu = 1, 2, \cdots$, exist so that the Hermite function $Je_n\{\eta\}$ can be formed. Suppose further that the series

$$f\{\eta\} = \sum_{n=0}^{\infty} Je_n\{\eta\} \tag{12.28}$$

converges for all functions $\eta(s)$ in the space $\mathfrak{R}$. We define the integral of the square of this functional $f\{\eta\}$ by

$$\int |\, f\{\eta\}\, |^2\, d(\theta\eta) = \sum_{n=0}^{\infty} \int |\, e_n(s)_n\, |^2\, dm(s)_n\,. \tag{12.29}$$

If the series (12.28) cannot be formed in the proper sense we nevertheless assign an ideal functional to it, and adopt for it also definition (12.29). The class of the real or ideal functionals for which (12.26) is finite will be denoted by $\mathfrak{F}$. For two functionals $f^{(1)}\{\eta\}$ and $f^{(2)}\{\eta\}$ of this class we define the inner product as

$$\int \bar{f}^{(1)}\{\eta\} f^{(2)}\{\eta\}\, d(\theta\eta) = \sum_{n=0}^{\infty} \bar{e}_n^{(1)}(s)_n e_n^{(2)}(s)_n\, dm(s)_n\,. \tag{12.30}$$

The definition (12.29) of the integral over the Hilbert space, is of course not quite satisfactory. One is tempted to call it opportunistic, so to say attained through the back door. It is naturally desirable to give a different definition of the integral in analogy to the definition of the ordinary integral as the limit of a sum, or at least to give a definition which is analogous to the definition of the integral over spaces of infinite dimension as developed in the general theory of measure. This will be done in a different publication.

In the theory of probability or of measure the integral is first assumed to be defined over cylinder spaces, see Kolmogoroff [20]. A cylinder space is determined by a set $\mathfrak{D}'$ in a finite dimensional subspace $\mathfrak{D}$ of the total space $\mathfrak{X}$; it consists of all vectors η in $\mathfrak{X}$ whose projection into $\mathfrak{D}$ lies in $\mathfrak{D}'$. The integral is then extended over a class of sets which can be built up in terms of cylinder spaces.

The integral over cylinder spaces can also be defined in the theory which we have discussed here. This can be done with the aid of Hermitian functionals on the basis of definition (12.29).[6] In fact, such an integral over the cylinder space has an important physical significance. It is the probability that the measurement of the quantities which have $\eta(x)$ as eigenvalues yield results having a projection into $\mathfrak{D}$ which lies in $\mathfrak{D}'$.

When the integral defined in this section is confronted with the integral as defined in measure theory a remarkable discrepancy appears. The space $\mathfrak{X}$ considered in probability theory contains the Hilbert space $\mathfrak{R}$ as a subspace.

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[6] See the thesis of J. Milkman, New York University, 1951 which contains a detailed mathematical justification of the statements made in this section and a number of additional facts.

When the integral defined in measure theory is extended over this Hilbert space $\mathfrak{K}$ or a subspace of it, the result is always zero. Thus it is seen that the two types of integral do not agree. The discrepancy can be resolved by the remark that the integral over cylinder spaces derived from definition (12.29) does not enjoy the property of total additivity, which is assumed in the theory of probability.[7]

We now return to our theory and discuss the possibility of representing states Φ by functionals $f\{\eta\}$.

In case the function $\eta(s)$ is given by a finite number of values $\eta_1, \cdots, \eta_r$ such that $\int \eta(s)^2\, dm(s) = \sum_{s=1}^{r} \eta_s^2$, the class $\mathfrak{F}$ consists just of the functions of $(\eta_1, \cdots, \eta_r)$ which are quadratically integrable and formula (12.28) holds for the value of the integral of $| f\{\eta\} |^2$. In this sense the integral defined by this formula comprises the integral over a finite dimensional space as a special case. Suppose a state Φ is represented by the set of functions $\psi_n(s)_n \equiv e_n(s)_n$. Then we consider the functional $f\{\eta\}$, given by (12.28), as a representer of the state Φ,

$$\Phi \leftrightarrow f\{\eta\}. \tag{12.31}$$

From relations (12.22), (12.23) we see that the annihilation and creation operators $A^{\mp}(s')$ are then represented by

$$A^{\mp}(s')\Phi \leftrightarrow \left[\frac{1}{2}\,\eta(s') \mp \frac{\delta}{\delta\eta(s')}\right] f\{\eta\}, \tag{12.32}$$

while the number operator $N = \int A^{+}(s)A^{-}(s)\, dm(s)$ is represented by

$$N\Phi \leftrightarrow \int \left[\frac{1}{2}\,\eta(s) - \frac{\delta}{\delta\eta(s)}\right]\left[\frac{1}{2}\,\eta(s) + \frac{\delta}{\delta\eta(s)}\right] dm(s) f\{\eta\}. \tag{12.33}$$

One easily verifies that the operators $A^{+}(s)$ and $A^{-}(s)$ represented by (12.32) obey the commutation rule (8.16).

It is now easy to define operators $\Xi(s)$, $\Pi(s)$ such that the representation $\Phi \leftrightarrow \phi(\xi)$ of the state Φ through the functional $\phi(\xi)$ implies the representations

$$\Xi(s)\Phi \leftrightarrow \xi(s)\phi(\xi), \tag{12.34}$$

$$\Pi(s)\Phi \leftrightarrow -i\,\frac{\delta}{\delta\xi(s)}\,\phi(\xi), \tag{12.35}$$

which, for $s = x$, agree with relations (1.3) and (2.4).

To this end we introduce the operator

$$Z^{S} = (2\Omega^{S})^{1/2}, \tag{12.36}$$

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[7] I owe this remark to B. Friedman.

Ω being the energy operator of a single particle, and set

$$\Xi(s) = [Z^s]^{-1}[A^+(s) + A^-(s)],$$

$$\Pi(s) = \frac{i}{2} Z^s [A^+(s) - A^-(s)],$$

in accordance with (8.29). Instead of the function $\eta(s)$ we introduce the function $\xi(s)$ through

$$\xi(s) = [Z^s]^{-1}\eta(s), \tag{12.37}$$

and instead of the functional $f\{\eta\}$ we introduce the functional

$$\phi(\xi) = f\{Z^s\xi\} = f\{Z\xi\}. \tag{12.38}$$

The representations (12.32) then go over into the representations (12.34), (12.35) as desired.

Relation (12.34), in particular, shows that the representation of the states Φ through the functionals $\phi(\xi)$ gives the simultaneous spectral representation of the operators $\Xi(s)$ assigned to all values s. The problem, formulated in Section 1, of finding a simultaneous spectral representation of operators $\Xi(s)$, $\Pi(s)$ (for $s = x$) which obey the commutation rule (8.30), is thus solved.

The expansion (12.28) of the functional $f\{\eta\}$ in terms of Hermitian functions leads to the expansion

$$\phi(\xi) = \sum_{n=0}^{\infty} Je_n\{Z\xi\} = \sum_{n=0}^{\infty} (n!)^{-1/2} He_n\{Z\xi\} \exp\{-(Z\xi, Z\xi)/4\}, \tag{12.39}$$

while the integral representation of the inner product will be written in the form

$$(\Phi^{(1)}, \Phi^{(2)}) = \int \bar{\phi}^{(1)}(\xi)\phi^{(2)}(\xi)\, d(\theta Z\xi). \tag{12.40}$$

Thus the class of functionals $\phi(\xi)$ which represent the states Φ is characterized.

The discussion given in this section appears to indicate that the representation of states Φ by functionals $\phi(\xi)$ is not the most suitable tool in the quantum theory of fields. In fact, this representation of states through functionals was not given directly; it was derived from the particle representation. It, therefore, appears that the particle representation is the proper tool to be used in order to put the mathematical notions and the mathematical arguments of field quantum theory on a firm mathematical foundation.

BIBLIOGRAPHY TO PARTS I AND II

Books

[1] Pauli, W., *Die allgemeinen Prinzipien der Wellenmechanik*, in *Handbuch der Physik*, Band XXIV/1, Springer, Berlin, 1933.

[2] Dirac, P. A. M., *The Principles of Quantum Mechanics*, Third Edition, Oxford University Press, 1947.

[3] Heitler, W., *The Quantum Theory of Radiation*, Second Edition, Oxford University Press, 1947.

[4] Wentzel, G., *Quantum Theory of Fields*, Interscience Publishers, New York, 1949.

Articles

[5] Born, M., and Jordan, P., *Zur Quantenmechanik*, Zeitschrift für Physik, Volume 34, 1925, pp. 858–888 (particularly pp. 883–887).

[6] Dirac, P. A. M., *On the theory of quantum mechanics*, Proceedings of the Royal Society of London, Series A, Volume 112, 1926, pp. 661–677.

[7] Heisenberg, W., *Mehrkörperproblem und Resonanz in der Quantenmechanik*, Zeitschrift für Physik, Volume 38, 1926, pp. 411–426.

[8] Dirac, P. A. M., *The quantum theory of the emission and absorption of radiation*, Proceedings of the Royal Society of London, Series A, Volume 114, 1927, pp. 246–265.

Dirac, P. A. M., *The quantum theory of dispersion*, Proceedings of the Royal Society of London, Series A, Volume 114, 1927, pp. 710–728.

[9] Jordan, P., and Pauli, W., *Zur Quantenelektrodynamik ladungsfreier Felder*, Zeitschrift für Physik, Volume 47, 1928, pp. 151–173.

[10] Jordan, P., and Wigner, E., *Über das Paulische Äquivalenzverbot*, Zeitschrift für Physik, Volume 47, 1928, pp. 631–651.

[11] Heisenberg, W., and Pauli, W., *Zur Quantendynamik der Wellenfelder*, Zeitschrift für Physik, Volume 56, 1929, pp. 1–61,

Heisenberg, W., and Pauli, W., *Zur Quantendynamik der Wellenfelder*, II, Zeitschrift für Physik, Volume 59, 1929, pp. 168–190.

[12] Landau, L., and Peierls, R., *Quantenelektrodynamik im Konfigurationsraum*, Zeitschrift für Physik, Volume 62, 1930, pp. 188–200.

[13] Fock, V., *Konfigurationsraum und zweite Quantelung*, Zeitschrift für Physik, Volume 75, 1932, pp. 622–647.

[14] Thomas, L. H., *Lecture notes in quantum electrodynamics*, Columbia University, 1947.

[15] Becker, R. and Leibfried, G., *Zur Methode der Quantisierung von Wellengleichungen*, Zeitschrift für Physik, Volume 125, 1948, pp. 347–358.

[16] Bohr, N., and Rosenfeld, L., *Field and charge measurement in quantum electrodynamics*, Physical Review, Volume 78, 1950, pp. 794–798.

See also: *Zur Frage der Messbarkeit der elektromagnetischen Feldgroessen*, Det Kgl. Danske Videnskabernes Selskab, Matematisk-fysiske Meddeleleser, Volume 12, No. 8, 1933.

Mathematical References

[17] Hilbert, D., *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Teubner, Leipzig, Berlin, 1924. First published in Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen, 1909–1910.

[18] Hellinger, E., *Die Orthogonalinvarianten quadratischer Formen von unendlich vielen Variabeln*, Dissertation, Göttingen, 1907.

[19] Von Neumann, J., *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin, 1932.

[20] Kolmogoroff, A. N., *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933.

[21] Rellich, F., *Der Eindeutigkeitssatz für die Lösungen der quantenmechanischen Vertauschungsrelationen*, Nachrichten der Akademie der Wissenschaften in Göttingen, Mathematisch-Physikalische Klasse, 1946.

22] Cameron, R. H., and Martin, W. T., *The orthogonal development of nonlinear functionals in series of Fourier-Hermite functionals*, Annals of Mathematics, Volume 48, 1947, pp. 385–389.

[23] Grad, H., *Note on N-dimensional Hermite polynomials*, Communications on Pure and Applied Mathematics, Volume II, No. 4, 1949, pp. 325–330.

[24] Schwartz, L., *Théorie des Distributions*, Hermann, Paris, 1950.

[25] Stone, M. H., *Linear transformations in Hilbert space and their applications to analysis*, American Mathematical Society Colloquium Publications, Volume XV, 1932.

[26] Nagy, Bela von Sz., *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Springer, Berlin, 1942.

Part III. Boson Field in Interaction with a Given Source Distribution

13. *Expectation Values of the Energy and the Number of Bosons*

The simplest specific problem that may be posed in field quantum theory is to determine what happens in a field under the influence of a given source distribution. In the subsequent discussion of this problem we shall employ the Heisenberg picture, unless otherwise stated. As before, the operator which corresponds to the "field quantity" will be denoted by Ξ and considered as a function $\Xi(x, t)$ of position x and time t. In case no external forces are present this function was supposed to satisfy the differential equation

$$\nabla_t^2\Xi + (\mu^2 - \nabla_x^2)\Xi = 0,$$

in which $\nabla_t = \partial/\partial t$, $\nabla_x = \text{grad}$, and μ is the "rest mass", see Section 3. In case the field is under the influence of a given distribution of charges with the density $j = j(x, t)$, the function $\Xi(x, t)$ should satisfy the differential equation

$$\nabla_t^2\Xi + (\mu^2 - \nabla_x^2)\Xi = j. \tag{13.1}$$

The term j on the right side is of course to be understood as the identity operator multiplied by the function $j(x, t)$.

The solution of this differential equation must satisfy initial conditions. In the present section we shall assume that the operators Ξ and $\nabla_t\Xi$ are prescribed at a finite time, specifically, at $t = 0$. Thus we require Ξ to satisfy condition

$$\Xi(x, 0) = \Xi_0(x), \qquad \nabla_t\Xi(x, 0) = \Pi_0(x), \tag{13.2}$$

in which $\Xi_0(x)$ and $\Pi_0(x)$ are two—improper—operators which obey the commutation rule

$$\Pi_0(x')\Xi_0(x'') - \Xi_0(x'')\Pi_0(x') = -i\delta(x' - x''). \tag{13.3}$$

In Section 16 we shall impose initial conditions which characterize the behavior of the operator $\Xi(x, t)$ as $t \to -\infty$.

The source distribution j will be assumed independent of the time,

$$j(x, t) = j(x)$$

in the present and in the two subsequent sections. In Section 16 we shall show that the results obtained can be used to deal with a general, time dependent, source distribution.

The "field quantity" Ξ may stand for various physical observables. If Ξ is a scalar it may be considered as the potential of a field of neutral mesons with the rest mass μ. But Ξ—and correspondingly also j—may just as well be a vector; for example, Ξ may be the four-component vector of an electromagnetic field. In this case the rest mass μ is zero, and in addition the "Lorentz condition" that the divergence of the potential vanish is imposed on Ξ. However, this condition need not concern us here. Finally, the observable Ξ may stand for the elongation in any elastic medium; this interpretation will play a special role in Part V.

Various significant operators can be formed in terms of $\Xi(x, t)$ and $\nabla_t\Xi(x, t)$. In so doing we shall employ the operator

$$\Omega^x = [\mu^2 - \nabla_x^2]^{1/2} \tag{13.4}$$

of the energy of a single particle. If the field is enclosed in a box, the expression (13.4) does not yet define the operator Ω completely; it is necessary to state which boundary conditions are imposed on the functions on which the operator Ω^x acts. We do not formulate such conditions explicitly.

With the aid of the operators Ξ, $\nabla_t\Xi$ and Ω we now form three operators: the operator

$$N(t) = \frac{1}{2}\int (\Omega^x\Xi - i\nabla_t\Xi)(\Omega^x)^{-1}(\Omega^x\Xi + i\nabla_t\Xi)\, dx, \tag{13.5}$$

corresponding to the "number of particles", cf. (10. 11), the operator

$$H(t) = \frac{1}{2}\int (\Omega^x\Xi - i\nabla_t\Xi)(\Omega^x\Xi + i\nabla_t\Xi)\, dx, \tag{13.6}$$

corresponding to the "intrinsic energy", cf. (10. 12), and the operator

$$V(t) = -\int \Xi j\, dx. \tag{13.7}$$

By formal differentiation, using the time independence of j, we derive from the differential equation (13.1) the relation

$$\nabla_t[H(t) + V(t)] = 0;$$

consequently,

$$H(t) + V(t) = \text{constant} \tag{13.8}$$

for any field satisfying equation (13.1) with $\nabla_t\, j = 0$. The constancy expressed in relation (13.8) justifies the interpretation of $H + V$ as the "total energy", and of V as the "interaction energy" except possibly for an additional constant. Reasons for giving this constant a specific value will appear at the end of this section. Meanwhile, we shall find it more convenient to disregard this constant and to use the term interaction energy for the observable which corresponds to the operator V as defined by (13.7).

The particles[1] which enter the description of the field are assumed to be *bosons*, i.e. the particle representers of the field are supposed to be symmetric.

The *main objective* of the present investigation is to determine both the expected values of the observables N, H, V at the time t, and the probabilities that the measurement of the observables N, H or $H + V$ at the time t yield given values. The limits of these expectation values and probabilities as the time t increases indefinitely are of particular interest. We assume that the state Φ for which these quantities are to be determined is given in such a way that they are known at the time $t = 0$. The case in which the state Φ is given by data referring to $t = -\infty$ will be treated in Section 16.

Two steps are needed in order to attain this objective: First we must find the solution $\Xi(x, t)$ of the differential equation (13.1) satisfying the initial conditions (13.2). This solution is easily found and the desired expectation values can then be readily calculated. The present Section will be devoted to carrying out this first step.

The second step consists in deriving the spectral representation of those observables for the measurements of which one desires to determine the probabilities.

Representations

If these observables were taken to be the field quantities $\Xi(x, t)$ it would be necessary to find a representation of the state by functionals $\phi(\xi, t)$, cf. Sections 1 and 12,

$$\Phi \underset{\xi}{\overset{t}{\leftrightarrow}} \phi(\xi, t),$$

such that the operators $\Xi(x, t)$ would be represented by

$$\Xi(x, t)\Phi \underset{\xi}{\overset{t}{\leftrightarrow}} \xi(x)\phi(\xi, t).$$

Such a representation could be found.

We are, however, more interested in N, H and other biquantized observables. Accordingly, we shall determine the particle representation

$$\Phi \underset{N}{\overset{t}{\leftrightarrow}} \{\psi_n(x)_n(t)\} \tag{13.9}$$

associated with the operator $N(t)$. We shall, of course, assume that the representers ψ_n are given at the time $t = 0$.

Incidentally, we shall deduce from the $N(t)$-representation that no scattering operator exists which connects the representation at $t = -\infty$ with that at $t = \infty$ in case the source distribution is constant in time.

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[1] In this exposition the term particle is used to designate entities with which a momentum K and a position X are associated as observables. Many quantum physical presentations use the term particle in a different sense, namely, to designate an entity with a rather well defined position.

The significance of the notion of position X of a boson is discussed in footnote 6.

In order to calculate the probabilities for the measurement of the observable $H + V$, a still different representation is to be employed. As mentioned above, the observable $H + V$ is the total energy of the field. Hence it remains constant in time, see (13.8). Accordingly, the spectral representation of the operator $H(t) + V(t)$ will yield a description of all stationary states of the field. Except for one remarkable case, this representation can be chosen as a particle representation. Consequently, we shall speak of "modified" particles; the number M of these modified particles remains constant in time.

The M- and $N(t)$-representations will be derived in Sections 14 and 15.

The exceptional case mentioned occurs if the rest mass μ is zero and, at the same time, the total charge $\int j(x)\,dx$ differs from zero. The peculiar occupation number representation needed for this case will be discussed in Part IV.

Operators and Differential Equation

It would not be difficult to solve the linear differential equation (13.1) by means of the same influence function that would be appropriate if $\Xi(x, t)$ were an ordinary function. This influence function is related to the function Γ discussed in Section 3. It is, however, much simpler to make use of unitary operators of the form $\exp\{\pm it\Omega\}$ for this purpose.

We shall also employ creation and annihilation operators $A^{\pm}(x, t)$ as a most convenient tool. These operators are connected with the operators $\Xi(x, t)$ and $\nabla_t\Xi(x, t)$ by the formula

$$A^{\pm}(x, t) = [\Omega^{X}/2]^{1/2}\Xi(x, t) \mp i[2\Omega^{X}]^{-1/2}\nabla_t\Xi(x, t), \tag{13.10}$$

see (10.10). Expressing Ξ and $\nabla_t\Xi$ in terms of $A^{\pm}$ in the differential equation (13.1) we find easily that this equation is equivalent with equation

$$\nabla_t A^{\pm}(x, t) \mp i\Omega^{X}A^{\pm}(x, t) = \mp i(2\Omega^{X})^{-1/2}j(x). \tag{13.11}$$

The initial condition (13.2) becomes

$$A^{\pm}(x, 0) = [\Omega^{X}/2]^{1/2}\Xi_0(x) \mp i[2\Omega^{X}]^{-1/2}\Pi_0(x), \tag{13.12}$$

the initial values $A^{\pm}(x, 0)$ obey the commutation law

$$A^{-}(x', 0)A^{+}(x'', 0) - A^{+}(x'', 0)A^{-}(x', 0) = \delta(x' - x''), \tag{13.13}$$

and the operators (13.4), (13.5), (13.6) become

$$\begin{aligned} N(t) &= \int A^{+}(x, t)A^{-}(x, t)\,dx, \\ H(t) &= \int A^{+}(x, t)\Omega^{X}A^{-}(x, t)\,dx, \\ V(t) &= -\int [A^{+}(x, t) + A^{-}(x, t)][2\Omega^{X}]^{-1/2}j(x)\,dx. \end{aligned} \tag{13.14}$$

We shall introduce a further modification which, though not necessary, will be helpful in the discussion of special questions.

Instead of the variable x we shall introduce a general quantum variable s with the measure function $m(s)$; s need not be connected with the variable x in a real manner.

The transform of any function $b(x)$ will be denoted by $b^-(s)$, the transform of its complex conjugate by $b^+(s)$. We omit the superscript when we write the absolute value,

$$|b^-(s)| = |b^+(s)| = |b(s)|. \tag{13.15}$$

We find it convenient to refer to the pair $b^\pm(s)$ as one function. We say the function $b^\pm(s)$ is quadratically integrable if $\| b \| < \infty$. Here

$$\| b \|^2 = (b, b) = \int |b(s)|^2 \, dm(s).$$

The manifold of all these functions forms the space $\mathfrak{K}$; see Section 6. The inner product of two functions $b_{1,2}(x)$ in $\mathfrak{K}$ is

$$(b_1, b_2) = \int \bar{b}_1(x) b_2(x) \, dx = \int b_1^+(s) b_2^-(s) \, dm(s). \tag{13.16}$$

We know that the annihilation operator $A^-(x)$, like the function $b(x)$, is transformed in the direct manner while the creation operator is transformed in the complex conjugate manner; see (8.24). Therefore, we have

$$\begin{aligned} (b, A) &= \int \bar{b}(x) A^-(x) \, dx = \int b^+(s) A^-(s) \, dm(s), \\ (A, b) &= \int A^+(x) b(x) \, dx = \int A^+(s) b^-(s) \, dm(s), \end{aligned} \tag{13.17}$$

using the convenient notations (b, A), (A, b) for the integrals (8.4) and (8.6).

We shall, furthermore, find it convenient to employ the functions

$$q^\pm(s) = [\Omega^S]^{-1} [2\Omega^S]^{-1/2} j^\pm(s). \tag{13.18}$$

The differential equations to be satisfied by the operators $A^\pm(s, t)$ can then be written in the form

$$\nabla_t A^\pm(s, t) \mp i\Omega^S A^\pm(s, t) = \mp i\Omega^S q^\pm(s), \tag{13.19}$$

as seen from (13.11). The initial conditions

$$A^\pm(s, 0) = A_0^\pm(s) \tag{13.20}$$

involve the transforms $A_0^\pm(s)$ of the operators $A^\pm(x, 0)$, see (13.12); they are supposed to satisfy the commutation rule

$$A_0^-(s') A_0^+(s'') - A_0^+(s'') A_0^-(s') = \delta(s' - s''). \tag{13.21}$$

The operators (13.14) become

$$N(t) = \int A^{+}(s, t)A^{-}(s, t)\, dm(s),$$

$$H(t) = \int A^{+}(s, t)\Omega^{s}A^{-}(s, t)\, dm(s), \tag{13.22}$$

$$V(t) = \int A^{+}(s, t)\Omega^{s}q^{-}(s)\, dm(s) - \int \Omega^{s}q^{+}(s)\cdot A^{-}(s, t)\, dm(s).$$

The last relation may simply be written as

$$V(t) = -(A(t), \Omega q) - (\Omega q, A(t)). \tag{13.23}$$

The differential equations (13.19) are now easily solved. To this end we introduce the operators

$$B^{\pm}(s, t) = A^{\pm}(s, t) - q^{\pm}(s), \tag{13.24}$$

so that equations (13.19) assume the form

$$\nabla_{t}B^{\pm}(s, t) \mp i\Omega^{s}B^{\pm}(s, t) = 0. \tag{13.25}$$

These equations are to be solved under the initial conditions

$$B^{\pm}(s, 0) = B^{\pm}_{0}(s) \tag{13.26}$$

with

$$B^{\pm}_{0}(s) = A^{\pm}_{0}(s) - q^{\pm}(s). \tag{13.27}$$

The solutions of equations (13.25) are evidently

$$B^{\pm}(s, t) = \exp\{\pm it\Omega^{s}\}B^{\pm}_{0}(s); \tag{13.28}$$

the solutions of equations (13.19) are therefore

$$A^{\pm}(s, t) = \exp\{\pm it\Omega^{s}\}[A^{\pm}_{0}(s) - q^{\pm}(s)] + q^{\pm}(s). \tag{13.29}$$

It is convenient to introduce the functions

$$q^{\pm}(t, s) = \exp\{\mp it\Omega^{s}\}q^{\pm}(s); \tag{13.30}$$

we then may write

$$A^{\pm}(s, t) = \exp\{\pm it\Omega^{s}\}[A^{\pm}_{0}(s) - q^{\pm}(s) + q^{\pm}(t, s)]. \tag{13.31$^{\pm}$}$$

We have not yet formulated any conditions for the function $j(x)$ describing the source distribution. We shall do so later on. At present we only mention that the right members of formulas (13.31)$^{\pm}$ define—improper—operators if the functions $q^{\pm}(s) - q^{\pm}(t, s)$ are quadratically integrable, i.e. belong to the space $\mathfrak{R}$. Setting

$$\| q - q(t) \|^{2} = \int | q(s) - q(t, s) |^{2}\, dm(s), \tag{13.32}$$

we express this condition as

$$\| q - q(t) \| < \infty. \tag{13.33}$$

It is certainly satisfied if the functions $q^{\pm}(s)$ themselves belong to $\mathfrak{R}$ or, setting

$$\| q \|^2 = \int | q(s) |^2 \, dm(s), \tag{13.34}$$

if

$$\| q \| < \infty. \tag{13.35}$$

There are, however, functions $q^{\pm}(s)$ which satisfy condition (13.33) but not (13.35). We shall, at first, ignore this fact, but later on we shall draw important conclusions from it.

With the aid of formulas (13.31)$^{\pm}$ we can express the operators $N(t)$, $H(t)$, $V(t)$ in terms of $A_0^{\pm}$ and $q^{\pm}$. Setting

$$\begin{aligned} N_0 &= N(0) = \int A_0^+(s) A_0^-(s) \, dm(s), \\ H_0 &= H(0) = \int A_0^+ \Omega^s A_0^-(s) \, dm(s), \\ V_0 &= V(0) = -(A_0 , \Omega j) - (\Omega j, A_0), \end{aligned} \tag{13.36}$$

we find, from (13.22) and (13.31),

$$\begin{aligned} N(t) = N_0 &- (A_0 , q - q(t)) - (q - q(t), A_0) \\ &+ (q - q(t), q - q(t)), \end{aligned} \tag{13.37}$$

$$\begin{aligned} H(t) = H_0 &- (A_0 , \Omega[q - q(t)]) - (\Omega[q - q(t)], A_0) \\ &+ (q - q(t), \Omega[q - q(t)]), \end{aligned} \tag{13.38}$$

$$\begin{aligned} V(t) = &-(A_0 , \Omega q(t)) - (\Omega q(t), A_0) \\ &+ (q - q(t), \Omega q(t)) + (\Omega q(t), q - q(t)). \end{aligned} \tag{13.39}$$

The right members of (13.38), (13.39) are defined if the function $\Omega^S q^{\pm}(s)$ belongs to $\mathfrak{R}$,

$$\| \Omega q \| < \infty; \tag{13.40}$$

for then $\Omega^S q^{\pm}(t, s)$ also belongs to $\mathfrak{R}$. Of course we assume that the function $q^{\pm}(s)$ satisfies condition (13.33).

If the function $q^{\pm}(s)$ itself belongs to $\mathfrak{R}$, i.e. satisfies condition (13.35), the expressions (13.37) to (13.39) can be written in the form

$$\begin{aligned} N(t) = N_0 - (A_0\,,\, q) - (q,\, A_0) + (A_0\,,\, q(t)) \\ + (q(t),\, A_0) + 2(q,\, q) - (q,\, q(t)) - (q(t),\, q), \end{aligned} \tag{13.41}$$

$$\begin{aligned} H(t) = H_0 + V_0 + (A_0\,,\, \Omega q(t)) + (\Omega q(t),\, A_0) \\ + 2(q,\, \Omega q) - (q(t),\, \Omega q) - (\Omega q,\, q(t)), \end{aligned} \tag{13.42}$$

$$\begin{aligned} V(t) = -(A_0\,,\, \Omega q(t)) - (\Omega q(t),\, A_0) + (q,\, \Omega q(t)) \\ + (\Omega q(t),\, q) - 2(q,\, \Omega q). \end{aligned} \tag{13.43}$$

Relation $H(t) + V(t) = H_0 + V_0$ is, of course, immediately verified from the last two formulas.

Asymptotic Expectation Values

We now proceed to evaluate the limits of the expected values of the observables N, H, V in any state Φ as the time t increases indefinitely.

Before determining the limits of these expectation values we shall formulate two lemmas. In agreement with (13.30) we shall use in these lemmas the abbreviation

$$b^{\pm}(t,\, s) = \exp\{\mp it\Omega^{s}\}b^{\pm}(s) \tag{13.44}$$

for any function $b(s)$ in $\mathfrak{K}$. We now formulate

Lemma 1. Suppose the field extends over the infinite space. Then

$$(b_1\,,\, b_2(t)) = \int b_1^{+}(s)b_2^{-}(t,\, s)\, dm(s) \to 0 \qquad \text{as} \qquad t \to \infty.$$

In order to prove the lemma we may choose the momentum k as the quantum variable s and write

$$(b_1\,,\, b_2(t)) = \int b_1^{+}(k)b_2^{-}(k) \exp\{it\omega\}\, dk.$$

Instead of $k = \{k_1\,,\, k_2\,,\, k_3\}$ one may introduce the variable $\omega = [\mu^2 + |\,k\,|^2]^{1/2}$ and two angular variables. The statement then follows from a well known theorem in the theory of Fourier transforms, namely the theorem that

$$\int_{\omega_0}^{\infty} \exp\{it\omega\}\phi(\omega)\, d\omega \to 0 \qquad \text{as} \qquad t \to \infty \tag{13.45}$$

for every integrable function ϕ of a real variable ω defined for $\omega_0 \leq \omega < \infty$.

Note that the statement of the lemma would not be true if the field were enclosed in a box. For, the energy ω would then depend on a label s with a discrete range of values and formula (13.45) would not be applicable.

Next we formulate

Lemma 2. Suppose the field extends over the infinite space. Let Φ be a state which admits the operator N_0. Then

$$(13.46) \qquad (\Phi, (A_0, b(t))\Phi) \to 0, \qquad (\Phi, (b(t), A_0)\Phi) \to 0 \quad \text{as} \quad t \to \infty.$$

In order to prove this statement we employ an N_0-representation of the state Φ,

$$\Phi \leftrightarrow \{\psi_n^-(k)_n\}.$$

Then, by (8.10),

$$(\Phi, (A_0, b(t))\Phi) = \sum_n n^{1/2} \int \psi_n^+(k)_n \psi_{n-1}^-(k)_{n-1} b(k_n) \exp\{it\omega_n\}\, d(k)_n .$$

By Lemma 1, each term in this series approaches zero as $t \to \infty$ since the function

$$a(k) = \int \psi_n^+(k)_{n-1}(k)\psi_{n-1}^-(k)_{n-1}\, d(k)_{n-1}$$

belongs to $\mathfrak{K}$. The infinite series can further be majorized by the series

$$\frac{1}{2}\sum_n n \,||\, \psi_n \,||^2 + \frac{1}{2}\sum_n ||\, \psi_{n-1} \,||^2 \,||\, b \,||^2$$

independently of t. Here the first series converges since Φ was required to admit the operator N_0. Consequently, the first statement (13.46) follows. The second statement (13.46) is proved in the same way.

We now use Lemma 2 to determine the limits of the expectation values of the observables N, H, V at the time t as $t \to \infty$. The *expectation value of an observable* L at the time t, corresponding to an operator $L(t)$, is defined by

$$(13.47) \qquad \langle L(t)\rangle_\Phi = (\Phi, L(t)\Phi)/(\Phi, \Phi).$$

We form the expectation values of $N(t)$, $H(t)$, $V(t)$ using expressions (13.41) to (13.43), and then derive from Lemma 2 the

Theorem: Suppose that the field extends over the infinite space and that, furthermore, $||\, q \,||$ and $||\, \Omega q \,||$ are finite. Let Φ be a state which admits the operators N_0, H_0, V_0. The expectation values of the observables N, H, V at the time t then approach limits as $t \to \infty$; these limits are given by

$$(13.48) \qquad \langle N(t)\rangle_\Phi \to \langle N_0\rangle_\Phi - \langle (A_0, q) + (q, A_0)\rangle_\Phi + 2(q, q),$$

$$(13.49) \qquad \langle H(t)\rangle_\Phi \to \langle H_0\rangle_\Phi + \langle V_0\rangle_\Phi + 2(q, \Omega q),$$

$$(13.50) \qquad \langle V(t)\rangle_\Phi \to -2(q, \Omega q).$$

We may interpret formulas (13.49) and (13.50) by saying: *The initial value of the interaction energy V is expected to be eventually absorbed in the intrinsic energy H; at the same time H is expected to rise by the amount* $2(q, \Omega q)$. *The interaction energy V, on the other hand, is expected to approach the negative value* $-2(q, \Omega q)$.

The quantity $2(q, \Omega q)$ has a simple significance. Using relation (13.18) to express q by j we find

$$(13.51) \qquad 2(q, \Omega q) = (j, \Omega^{-2} j).$$

Since $(\Omega^X)^2 = \mu^2 - \nabla_x^2$, it is clear that $-(j, \Omega^{-2}j)$ is simply Coulomb's interaction energy, in case $\mu = 0$, and Yukawa's interaction energy in case $\mu > 0$. Specifically, $(\Omega^X)^{-2}$ is the integral operator with the kernel

$$[4\pi \mid x' - x'' \mid]^{-1} \exp \{-\mu \mid x' - x'' \mid\},$$

as is well known. Therefore, the expression $(j, \Omega^{-2}j)$ can be written in the familiar form

$$(13.52) \qquad \begin{aligned} &(j, \Omega^{-2}j) \\ &= \iint j(x')[4\pi \mid x' - x'' \mid]^{-1} \exp \{-\mu \mid x' - x'' \mid\} j(x'')\, dx'\, dx''. \end{aligned}$$

The above interpretation of formulas (13.49) and (13.50) can now be restated: *The interaction energy V is expected to approach the Coulomb-Yukawa interaction energy $-(j, \Omega^{-2}j)$. The intrinsic energy is expected to rise by an amount equal to the difference of the initial and the final expected value of V.*

The interaction energy refers, of course, to the interaction between the distributed sources and the boson field. These sources also exert forces on each other, but indirectly, with the field as intermediary. Since the field quantity Ξ does not explicitly enter the expression $-(j, \Omega^{-2}j)$ for the asymptotic expectation value of the interaction energy, this asymptotic value may be interpreted as the energy of the interaction of the sources among themselves.

In the case of an electromagnetic field subject to the influence of distributed charged matter, the Coulomb energy thus results as the energy of interaction. In case of a meson field subjected to the influence of nucleons, according to Yukawa's theory, Yukawa's energy results as the energy of interaction between nucleons. It should be mentioned that the changes in the source distribution which results from this interaction are neglected in the formulation of the problem adopted in this section.

In discussing the significance of formula (13.48) it is convenient to assume that the state Φ is such that the number N of its particles is determined initially. In other words, we assume that of the sequence of N_0-representers of Φ only one, $\psi_n(s)_n$, differs from zero, so that we may write $\Phi = \Phi_n$. The operators A_0^+ produce zero from such a state; hence formula (13.48) can be written as

$$(13.53) \qquad \langle N(t)\rangle_{\Phi_n} \to n + 2(q, q) \qquad \text{as} \qquad t \to \infty.$$

In other words, *the number of particles is expected to rise by* $2 \,||\, q \,||^2$, *independently of the initial number n.* The numbers $2 \,||\, q \,||^2$ and $2(q, \Omega q)$ will be called the "number increment" and the "energy increment", respectively, produced by the source distribution $j(x)$.

Note that this increase of the number of particles takes place even if $n = 0$, i.e. if in the state $\Phi = \Phi_0$ the field was in a vacuum at the time $t = 0$. This phenomenon of the creation of particles independently of the number initially present has been called *spontaneous emission.*

Infrared Catastrophe

If the increment $2 \| q \|^2$ is infinite, formula (13.53) shows that the number of particles rises indefinitely. This phenomenon has been called an "infrared catastrophe", because, as we shall see below, it is due to the behavior of the transform $j^{\sim}(k)$ of the source distribution $j(x)$ near $k = 0$, in other words, it is caused by low wave numbers. More precisely we formulate the statement:

In case $\| q \|$ *is infinite but* $\| q - q(t) \|$ *is finite, the expected number of particles* $\langle N(t) \rangle_{\Phi_n}$ *is finite for every finite time t but approaches infinity as* $t \to \infty$.

To justify this statement we first infer from (13.37) the relation

$$\langle N(t) \rangle_{\Phi_n} = n + \| q - q(t) \|^2.$$

Clearly, with $\omega = \omega(k)$,

$$\| q - q(t) \|^2 = 2 \int [1 - \cos \omega t] \mid q(k) \mid^2 dk.$$

Hence

$$\| q - q(t) \|^2 \leq t^2 \int \omega^2 \mid q(k) \mid^2 dk = t^2 \| \Omega q \|^2 < \infty.$$

This formula indicates that the failure of $\| q \|$ to be finite although $\| \Omega q \|$ is finite results from the behavior of $q^{\sim}(k)$ at those values of k for which $\omega(k) = 0$. Excluding a neighborhood of $\omega = 0$ we find

$$\| q - q(t) \|^2 \geq 2 \int_{\omega \geq \epsilon} [1 - \cos \omega t] \mid q(k) \mid^2 dk.$$

Here the right member approaches $2 \int_{\omega \geq \epsilon} \mid q(k) \mid^2 dk$ as $t \to \infty$, and this intergal can be made arbitrarily large by making ϵ sufficiently small. Thus the above statement follows.

Obviously, $\omega(k)$ can vanish only for $\mu = 0$, and if μ equals zero, $\omega(k)$ vanishes for $k = 0$. It is thus seen that the catastrophe in which infinitely many particles are emitted is caused by the behavior of $j^{\sim}(k)$ near the small "infrared" wave numbers k.

In order to investigate in greater detail under which circumstances the increment $2 \| q \|^2$ is finite we express it in terms of j by (13.18). We find

$$2(q, q) = (j, \Omega^{-3} j) \tag{13.54}$$

and, using k as quantum variable,

$$(j, \Omega^{-3} j) = \int [\mu^2 + \mid k \mid^2]^{-3/2} \mid j(k) \mid^2 dk. \tag{13.55}$$

Condition (13.35), that $\| q \|$ be finite, thus means that the integral on the right side of relation (13.55) should be finite.

When $\mu > 0$ this condition is satisfied if the function $j^{\sim}(k)$ dies out sufficiently strongly as $\mid k \mid \to \infty$; this means that the singularities of the function $j(x)$ may not be too severe.

When $\mu = 0$, however, the condition $\| q \| < \infty$ implies a severe restriction at $k = 0$. If $j^*(k)$ is continuous at $k = 0$, condition (13.35) requires that $j^*(k)$ vanish at $k = 0$

$$j^*(k) = 0 \quad \text{at} \quad k = 0. \tag{13.56}$$

For the function $j(x)$ this means that

$$\int j(x)\, dx = 0, \tag{13.57}$$

or, in other words, it means that the *total charge be zero.*

Without giving details, we mention that relation (13.57), if properly interpreted, is necessary for $\| q \| < \infty$, even if $j^*(k)$ is not continuous at $k = 0$. Therefore, we may say that *in case the rest mass μ is zero the number of particles is expected to rise by a finite amount if the total charge vanishes and by an infinite amount if it does not.*

It may be of some interest to have the increment $2(q, q)$ expressed in terms of the charge density $j(x)$ when $\mu = 0$ and $\int j(x)\, dx = 0$. The expression is

$$2(q, q) = -2(2\pi)^{-2} \iint j(x') \log | x' - x'' | j(x'')\, dx'\, dx'' \quad \text{if} \quad \mu = 0. \tag{13.58}$$

It is to be noted that this value does not change if a constant is added to the logarithm as seen from (13.57). It should be emphasized that the relation $\| q \| = \infty$ does not exclude the possibility that the energy increment $2(q, \Omega q)$ is finite. In other words, although the number of particles is expected to become infinite in an infrared catastrophe, the total energy carried by these infinitely many particles may still be expected to be finite.

We mention that the infrared catastrophe has been investigated by Bloch and Nordsiek [29], in a more general manner, taking into account the back reaction of the field on the sources.

We shall summarize conditions (13.40), (13.33), (13.35) imposed on the function $q^*(s)$ in order to avoid the catastrophe. We express these conditions in terms of the transforms $j^*(k)$ of the charge distribution $j(x)$, assuming the field to extend over the infinite space. They now read

$$2 \| \Omega q \|^2 = \int \omega^{-1} | j(k) |^2\, dk < \infty, \tag{13.59}$$

$$\| q - q(t) \|^2 = \int (1 - \cos \omega t)\omega^{-3} | j(k) |^2\, dk < \infty, \tag{13.60}$$

and

$$2 \| q \|^2 = \int \omega^{-3} | j(k) |^2\, dk < \infty, \tag{13.61}$$

with

$$\omega = [\mu^2 + | k |^2]^{1/2}.$$

We note, for example, that a point charge e at the point x_0, given by

$$j(x) = e\delta(x - x_0) \tag{13.62}$$

leads to a transform

$$j^{\mp}(k) = e \exp \{\mp ikx_0\}, \tag{13.63}$$

which does not satisfy any of these three conditions. Accordingly, point charges are excluded.

Modified Particles

We proceed to discuss the notion of *modified particles.* In the process of solving the basic differential equations (13.19) we introduced the operators $B^{\mp}(s, t)$, see (13.24). It is immediately evident that these operators also obey the commutation law

$$B^-(s', t)B^+(s'', t) - B^+(s'', t)B^-(s', t) = \delta(s' - s''), \tag{13.64}$$

characteristic for creation and annihilation operators. One is, therefore, inclined to conjecture that these operators are creation and annihilation operators with respect to an appropriate modified particle representation. In Section 14 we shall show that this is indeed the case provided $\| q \| < \infty$. But even if $\| q \| = \infty$ it is possible to consider the operators $B^{\mp}$ as creation and annihilation operators; however, these operators are then no longer associated with a particle representation. This will be shown in Section 17, Part IV.

Assuming $\| q \| < \infty$ we shall prove in Section 14 the existence of a representation

$$\Phi \underset{M}{\leftrightarrow} \{\chi_m(s)_m(t)\} \tag{13.65}$$

of the state Φ in terms of functions $\chi_m(s)_m = \chi_m(s_1 , \cdots , s_m)$ such that the states $B^{\mp}(s', t)\Phi$ are represented by

$$\begin{aligned} B^-(s', t)\Phi &\underset{M}{\leftrightarrow} \{(m + 1)^{1/2}\chi^-_{m+1}(s)_m(s')(t)\}, \\ B^+(s', t)\Phi &\underset{M}{\leftrightarrow} \{m^{1/2}Sy\, \chi^-_{m-1}(s)_{m-1}(t)\, \delta(s' - s_m)\}. \end{aligned} \tag{13.66}$$

We can now introduce *modified* biquantized operators, in particular the *modified number*

$$M(t) = \int B^+(s, t)B^-(s, t)\, dm(s) \tag*{13.67)}$$

and the *modified energy*

$$H_{\text{tot}}(t) = \int B^+(s, t)\Omega^S B^-(s, t)\, dm(s). \tag{13.68}$$

From relation (13.28),

$$B^{\mp}(s, t) = \exp \{\pm it\Omega^S\}B^{\mp}_0(s),$$

we now deduce that these operators are constant in time

$$M(t) = M \quad = \text{constant}, \tag{13.69}$$

$$H_{\text{tot}}(t) = H_{\text{tot}} = \text{constant}. \tag{13.70}$$

Moreover, it follows from relation (13.28) that if we set $\chi_m^-(s)_m(0) = \chi_m^-(s)_m$ the representers $\chi_m(s)_m(t)$ can be so chosen that

$$\chi_m^-(s)_m(t) = \exp\{-it\Omega^S\}\chi_m^-(s)_m . \tag{13.71}$$

Relation (13.71) makes it clear that the states with the initial representers

$$\chi_m^-(k)_m = \delta_{mm'}\, \delta(k_1 - k_1') \cdots \delta(k_{m'} - k_{m'}')$$

are the stationary states in which the field consists of m' modified particles with the momenta $k_1', \cdots, k_{m'}'$. Representation (13.65) together with (13.71) is therefore called a representation in terms of "stationary states."

The relationship between the modified and unmodified number operators is evidently

$$M = N(t) - (A(t), q) - (q, A(t)) + (q, q), \tag{13.72}$$

$$N(t) = M + (B(t), q) + (q, B(t)) + (q, q), \tag{13.73}$$

as seen from (13.24). The relationship between the modified and unmodified energies is

$$H_{\text{tot}} = H(t) - (A(t), \Omega q) - (\Omega q, A(t)) + (q, \Omega q), \tag{13.74}$$

$$H(t) = H_{\text{tot}} + (B(t), \Omega q) - (\Omega q, B(t)) + (q, \Omega q). \tag{13.75}$$

Comparing (13.74) with (13.23) we find relation

$$H_{\text{tot}} = H(t) + V(t) + (q, \Omega q). \tag{13.76}$$

It thus appears that the observable $V + (q, \Omega q)$, instead of V, should have been called "interaction energy" in order that the total energy could be obtained by a biquantization process from Einstein's expression $\Omega^K = [\mu^2 + |k|^2]^{1/2}$ for the energy of a single particle.

Suppose the state Φ is such that the field consists of just n particles at the time $t = 0$; i.e. $\Phi = \Phi_n$. Then, as seen from (13.72), the expected number of modified particles is

$$\langle M \rangle_{\Phi_n} = n + (q, q). \tag{13.77}$$

This number is, therefore, larger than the number n of unmodified particles; the difference between the two equals one half the expected eventual increase of the latter number. Relation (13.77) together with the limit relation for $t \to \infty$ in case the field extends over the infinite space may be written in the form

$$(13.78)\qquad \begin{aligned}\langle N(0)\rangle_{\Phi_n} &= \langle M\rangle_{\Phi_n} - (q, q),\\ \langle N(\infty)\rangle_{\Phi_n} &= \langle M\rangle_{\Phi_n} + (q, q),\end{aligned}$$

making use of (13.53) and the constancy of M.

If the field is in a state $\Phi = \Phi'_m$ in which it consists of m modified particles, we use (13.73) and find

$$(13.79)\qquad \langle N(t)\rangle_{\Phi'_m} = m + (q, q);$$

in words, the number of unmodified particles in the state Φ'_m is expected to be larger than the number m by the amount (q, q), independently of the time t.

Similar facts could of course also be formulated for the energies H, V, and H_{tot}. From (13.75) we find

$$(13.80)\qquad \langle H(t)\rangle_{\Phi'_m} = \langle H_{\text{tot}}\rangle_{\Phi'_m} + (q, \Omega q);$$

hence, from (13.76),

$$(13.81)\qquad \langle V(t)\rangle_{\Phi'_m} = -2(q, \Omega q).$$

Note that these expectation values are independent of m; also, they do not vary in time. The expectation value of $V(t)$ is in particular the same as the asymptotic expectation value found earlier, cf. (13.50). Therefore, this asymptotic value is already attained initially if the state is an eigenstate Φ'_m of the modified number M.

In this connection we mention the fact, to be proved in Section 16, that the eigenstates Φ'_m of the modified number can be produced from the corresponding eigenstates Φ_m of the unmodified number N by switching on the source distribution $j(x)$ infinitely slowly.

14. *Particle Representation of the States of the Boson Field Modified by a Source Distribution*

In Section 13 we anticipated that the states Φ of the boson field would admit representations of the type of particle representation associated with the modified operators

$$(14.1)\qquad B^{*}(s, t) = A^{*}(s, t) - q^{*}(s).$$

We want to construct such a representation. Suppose we know the representations

$$(14.2)\qquad \Phi \underset{N}{\overset{t}{\leftrightarrow}} \{\psi_n(s)_n(t)\}$$

of the state of the field, i.e. the $N(t)$-representations, associated with the creation and annihilation operators $A^{*}(s, t)$. Then we want to produce a representation

$$(14.3)\qquad \Phi \underset{M}{\overset{t}{\leftrightarrow}} \{\chi_m(s)_m(t)\}$$

associated with the operators $B^*(s, t)$. Vice versa, the representers ψ_n must, of course, be expressed in terms of the χ_n. We shall then be able to compute the probabilities that m modified particles will be found in the field consisting of n unmodified particles and vice versa.

Our second problem will be to find the representers $\psi_n(s)_n(t)$ at any time in terms of the representers $\psi_n(s)_n(0)$ at the initial time $t = 0$, and vice versa. We may also say that we want to express the representation associated with the operators

$$A^*(s, t) = \exp\{\pm it\Omega^S\}[A_0^*(s, 0) + q^*(s) - q^*(t, s)], \tag{14.4}$$

see (13.29), in terms of the representation associated with the operators $A_0^*(s)$ and vice versa. Having done so we shall be able to compute the "transition" probabilities, such as the probability that the state will consist of n particles at the time t if it consisted of n_0 particles at the time $t = 0$.

We shall also compute the limits of these probabilities as $t \to \infty$. We shall find that these limits are the same if $t \to -\infty$; hence the over-all process cannot be described by a scattering operator in the strict sense.

The situation may be quite different if the source distribution is not constant in time. A scattering operator does exist if the sources $j = j(x, t)$ are switched on in such a way that $j(x, t) \to 0$ as $t \to -\infty$ and eventually switched off again in such a way that $j(x, t) \to 0$ as $t \to \infty$. In the general case of a time dependent source distribution, the annihilation and creation operators can be brought into the form

$$A^*(s, t) = \exp\{\pm it\Omega^S\}[A_0^*(s) - g^*(s, t)] \tag{14.5}$$

with an appropriate function $g^*(s, t)$, as will be shown in Section 16.

Relations (14.4) and (14.5) are of the same type as relation (14.1) except for the exponential factor. Consequently, all three problems can be solved in the same way. The first problem will be solved in the present section; the second and third problems will be solved in Sections 15 and 16.

Modified Particle Representation

In solving the first problem we may omit all reference to the time and ask for the transformation of the representation

$$\Phi \underset{N}{\leftrightarrow} \{\psi_n(s)_n\} \tag{14.6}$$

associated with operators $A^*(s)$ into a representation

$$\Phi \underset{M}{\leftrightarrow} \{\chi_m(s)_m\} \tag{14.6$'$}$$

associated with operators

$$B^*(s) = A^*(s) - g^*(s), \tag{14.7}$$

and vice versa, see (14.1).

For the sake of generality, we have introduced here functions $g^*(s)$ instead of $q^{\pm}(s)$. We assume that the functions $g^{\pm}(s)$ belong to the space $\mathfrak{R}$; i.e. we assume

$$(14.8) \qquad ||\, g \,||^2 = (g, g) = \int |\, g(s) \,|^2 \, dm(s) = \int g^+(s) g^-(s) \, dm(s) < \infty .$$

We do not require the function $g^{\pm}(s)$ to be real when the position variable x is taken as the quantum variable s. Of course, both the functions $\psi_n(s)_n$ and the functions $\chi_n(s)_n$ should belong to the class $\mathfrak{R}_n$; also relation

$$(14.9) \qquad (\Phi, \Phi) = \sum_n ||\, \psi_n \,||^2 = \sum_n ||\, \chi_n \,||^2$$

is implied.

Suppose a transformation of the representer ψ_n into the representer χ_n exists. Since these representers constitute the same class of functions this transformation can be expressed by means of an operator.

We recall that at the end of the introduction the term operator was reserved for a mapping of the elements of a space into elements of the same space, and the term transformation for the mapping of the representers of elements into other representers of the same elements.

The desired transformation will be denoted by T_M^N, so that relation

$$(14.10) \qquad \chi_m = T_M^N \psi_n$$

holds. For convenience, the variables $(s)_n$ and $(s)_m$ have been omitted in (14.10). The inverse of T_M^N will be denoted by T_N^M ; thus

$$(14.10)^* \qquad \psi_n = T_N^M \chi_m \, .$$

Let L be any operator acting on the states Φ and let L^N, L^M be the functional operators which represent L when Φ is represented by ψ_n and χ_n respectively; then, clearly,

$$(14.11) \qquad L^M T_M^N = T_M^N L^N .$$

In particular we may choose the operators $A^{\pm}(s')$, $B^{\pm}(s')$ for L and denote their representers by $A_N^{\pm}(s')$, $B_N^{\pm}(s')$ and $A_M^{\pm}(s')$, $B_M^{\pm}(s')$. We then have

$$(14.12) \qquad B_M^{\pm}(s') T_M^N = T_M^N B_N^{\pm}(s') .$$

The functional operator $A_N^{\pm}(s')$ has the familiar form

$$(14.13) \qquad \begin{aligned} A_N^-(s')\{\psi_n(s)_n\} &= \{[n+1]^{1/2} \psi_{n+1}(s)_n(s')\}, \\ A_N^+(s')\{\psi_n(s)_n\} &= \{[n]^{1/2} \, Sy \, \psi_{n-1}(s)_{n-1} \, \delta(s' - s_n)\} . \end{aligned}$$

Note that the operator $A_N^{\pm}(s')$ here acts on ψ as a function of $n, s_1, s_2, \cdots$ and in turn produces another such function; $A_N^{\pm}(s')$ does not denote the transformation of the n-th component ψ_n of Φ into the $(n \pm 1)$-st component of $A^{\pm}(s')\Phi$.

The operators $B_N^{\pm}(s')$ are, of course, given by

$$(14.14) \qquad B_N^{\pm}(s') = A_N^{\pm}(s') - g^{\pm}(s') .$$

We now desire to determine the transformation T^N_M and its inverse T^M_N in such a way that the operators

$$T^N_M B^{\pm}_N(s') T^M_N$$

are the same as the operators $B^{\pm}_M(s')$ characterized by

$$\begin{aligned} B^-_M(s')\{\chi_m(s)_m\} &= \{[m+1]^{1/2}\chi_{m+1}(s)_m(s')\}, \\ B^+_M(s')\{\chi_m(s)_m\} &= \{m^{1/2} Sy\, \chi_{m-1}(s)_{m-1}\, \delta(s' - s_m)\}. \end{aligned} \tag{14.15}$$

Instead of determining T^M_N directly, we shall first determine an operator T which transforms a state Φ with the N-representers ψ into a state $T\Phi$ whose N-representers are the same functions χ that serve as the M-representers of the state Φ. We may express this property by

$$T^N \psi_n = \chi_n\,. \tag{14.16}$$

In order to connect the operator T with the transformation T^N_M we introduce the "trivial" transformation I^M_N which renames the label m into n, i.e. which transforms the functions χ_m into the functions χ_n,

$$I^M_N \chi_m = \chi_n\,. \tag{14.17}$$

The operator T^N can then be written in the form

$$T^N = I^M_N T^N_M\,. \tag{14.18}$$

Clearly, since by (14.15), (14.13) the operators $B^{\pm}_M(s')$ act on the functions χ_m precisely as the operators $A^{\pm}_N(s')$ act on the functions ψ_n, relation (14.17) entails relation

$$A^{\pm}_N(s') I^M_N = I^M_N B^{\pm}_M(s'). \tag{14.19}$$

Combining this relation with relations (14.12) and (14.18) we find that T^N satisfies relation

$$A^{\pm}_N(s') T^N = T^N B^{\pm}_N(s'); \tag{14.20}$$

the latter is equivalent with relation

$$A^{\pm}(s') T = T B^{\pm}(s'), \tag{14.21}$$

which does not refer to any representation.

We maintain that *the operator T is unitary*. For, since ψ_n and $\chi_m = T^N_M \psi_n$ represent the same state, we have $\sum_m \| \chi_m \|^2 = \sum_n \| \psi_n \|^2$. On the other hand, it is evident that $\sum_n \| \chi_n \|^2 = \sum_m \| \chi_m \|^2$ and thus we have $\sum_n \| \chi_n \|^2 = \sum_n \| \psi_n \|^2$. In other words, the operator T is length preserving and since T^N_M and I^M_N have inverses, the same is true of T; therefore T is unitary.

Conversely, suppose we have found a unitary operator T which satisfies relation (14.21). The transformation

$$T^N_M = I^N_M T^N \tag{14.22}$$

defined with the aid of the inverse I^N_M of I^M_N then satisfies relation (14.10) and is therefore the desired transformation of the N-representation into the M-representation of the states Φ.

The method of reduction of a transformation to an operator, sometimes called the method of "canonical transformation", is, if possible, a most handy procedure. It is always possible if the representers to be transformed into each other form the same class of functions. Otherwise different, less elegant, methods must be used. In fact, even if the functions $g^{*}(s)$ do not belong to $\mathfrak{K}$, i.e. if $\| g \| = \infty$, a simultaneous spectral representation of all the operators which result through biquantization can still be found with the aid of the operators B^{*}, see Section 19. A particle representation is then no longer possible; therefore, the method of canonical transformation is not applicable to this case.

First Form of the Operator T

A unitary operator T which satisfies relation (14.21) can easily be given explicitly in several forms. We first mention the form

$$T = T\{g\} = \exp\{(g, A) - (A, g)\}, \tag{14.23}$$

although we shall not use it later on. Here we have employed the abbreviation

$$\begin{aligned} (g, A) &= \int g^{+}(s)A^{-}(s)\, dm(s), \\ (A, g) &= \int A^{+}(s)g^{-}(s)\, dm(s), \end{aligned} \tag{14.24}$$

in agreement with (13.18). The operators (g, A) and (A, g) are evidently adjoint to each other. They can be shown to be applicable to the same manifold of states Φ. Therefore, the operator $L = -i(g, A) + i(A, g)$ is self-adjoint, and the operator $\exp\{iL\}$ is defined, as is well known. At the same time it follows that the operator $T\{g\}$ is unitary. In fact, its adjoint,

$$T^{*} = T\{-g\} = \exp\{(A, g) - (g, A)\}, \tag{14.25}$$

is evidently its inverse,

$$T^{-1} = T^{*}. \tag{14.26}$$

Our next task is to verify that the operator $T = T\{g\}$ satisfies relation (14.21). We could do this by employing a power series expansion[2] of $T\{g\}$,

$$T\{g\} = \sum_{r} r\text{!`}\{(g, A) - (A, g)\}^{r},$$

and then evaluating the expression

$$A^{*}(s')\{(g, A) - (A, g)\}^{r} - \{(g, A) - (A, g)\}^{r}B^{*}(s').$$

---

[2] In the following we shall use the abbreviation

$$\begin{aligned} r\text{!`} &= [r!]^{-1} && \text{for} \quad r = 0, 1, \cdots \\ &= 0 && \text{for} \quad r = -1, -2, \cdots \end{aligned}$$

for the reciprocal factorial.

Instead, we shall proceed in a manner customary in the theory of continuous groups. We shall consider the operator $T\{\alpha\, g\}$ in which α is a variable number, and form the operators

$$P^{\pm}(\alpha) = T\{-\alpha g\}A^{\pm}(s')T\{\alpha g\} - A^{\pm}(s') + \alpha g^{\pm}(s').$$

The derivatives of these operators with respect to α are obviously

$$\frac{d}{d\alpha} P^{\pm}(\alpha) = -T\{-\alpha g\}[\{(g, A) - (A, g)\}, A^{\pm}(s')]T\{\alpha g\} + g^{\pm}(s')$$

where the customary commutator abbreviation

$$[L_1\, ; L_2] = L_1L_2 - L_2L_1 \tag{14.27}$$

has been used. From the commutation rules (8.16) we deduce

$$[\{(g, A) - (A, g)\}, A^{\pm}(s')] = g^{\pm}(s')$$

and hence we have

$$\frac{d}{d\alpha} P^{\pm}(\alpha) = 0.$$

We conclude that

$$P^{\pm}(1) = P^{\pm}(0) = 0.$$

In particular, the relation $P^{\pm}(1) = 0$ is exactly the relation (14.21) which was to be derived, as seen from (14.7).

To make this reasoning rigorous certain additional arguments are of course needed. These we shall not give; but assuming that they can be supplied we may say that *with the explicit construction of the operator T we have established the existence of a representation, the M-representation, for which the operators $B^{\pm}(s)$ are creation and annihilation operators.*

We have presented the operator $T\{g\}$ as the operator T to be found and verified that it has the desired properties; we have not offered any motivation that might have led to the expression $T\{g\}$. This we shall now do.

We shall assume that the functions $g^{\pm}(s) = q^{\pm}(s)$ are real when the position x is taken as the quantum variable; we then can form the Hermitian operators

$$\Xi(x) = [2\Omega^{X}]^{-1/2}[A^{+}(x) + A^{-}(x)],$$

$$\Pi(x) = \frac{i}{2}\,[2\Omega^{X}]^{1/2}[A^{+}(x) - A^{-}(x)].$$

In terms of the operators $B^{\pm}(x) = A^{\pm}(x) - q(x)$ we form the operator

$$\mathrm{H}(x) = [2\Omega^{X}]^{-1/2}[B^{+}(x) + B^{-}(x)]$$

which plays the rôle of a "modified field quantity," while $\Pi(x)$ plays the same rôle for $A^{\pm}(x)$ as for $B^{\pm}(x)$. Clearly, the operators H and Ξ are connected by the relation

$$\mathrm{H}(x) = \Xi(x) - \hat{\xi}(x)$$

with

$$\hat{\xi}(x) = [\Omega^X/2]^{-1/2}q(x).$$

Suppose a representation of the states Φ by functionals $f(\xi)$ of functions $\xi(x)$ is known in such a way that

$$\Phi \underset{\xi}{\leftrightarrow} f(\xi)$$

implies

$$\Xi(x)\Phi \underset{\xi}{\leftrightarrow} \xi(x)f(\xi).$$

Then the representation

$$\Phi \underset{\eta}{\leftrightarrow} f(\eta + \hat{\xi})$$

of Φ by functionals of $\eta(x)$ evidently implies the representation

$$\mathrm{H}(x)\Phi \underset{\eta}{\leftrightarrow} \eta(x)f(\eta + \hat{\xi}).$$

Thus the η-representer is found from the ξ-representer by a translation. In keeping with a formula used in certain branches of operational calculus we may try to represent this translation by the exponential function of a differentiation,

$$f(\eta + \hat{\xi}) = \exp\{\hat{\xi}\delta/\delta\eta\}f(\eta)$$

or, replacing η by ξ, by

$$f(\xi + \hat{\xi}) = \exp\{\hat{\xi}\delta/\delta\xi\}f(\xi).$$

Here the exponential operator transforms the ξ-representer $f(\xi)$ into the η-representer $f(\eta + \hat{\xi})$ with ξ substituted for η. It is thus indicated that this operator is the representer of the operator T,

$$T^\xi = \exp\{\hat{\xi}\delta/\delta\xi\}.$$

As we know from (2.4) and (12.35), the functional operator $-i\delta/\delta\xi$ is the representer of the operator $\Pi(x)$,

$$\Pi(x)\Phi \underset{\xi}{\leftrightarrow} -i\frac{\delta}{\delta\xi}f(\xi).$$

Consequently, $T^\xi = \exp\{\hat{\xi}\delta/\delta\xi\}$ is the ξ-representer of the operator

$$T = \exp\{i\hat{\xi}\Pi\}.$$

The inner product $\hat{\xi}\Pi$ is simply

$$\begin{aligned}\hat{\xi}\Pi &= \int \hat{\xi}(x)\Pi(x)\,dx = \frac{i}{2}\int \hat{\xi}(x)[2\Omega^X]^{1/2}[A^+(x) - A^-(x)]\,dx \\ &= \frac{i}{2}\int [2\Omega^X]^{1/2}\hat{\xi}(x)\cdot[A^+(x) - A^-(x)]\,dx \\ &= i\int q(x)[A^+(x) - A^-(x)]\,dx\end{aligned}$$

or

$$i\hat{\xi}\Pi = (q, A) - (A, q).$$

Thus the relation $\exp \{i\hat{\xi}\Pi\} = T\{q\}$ is derived.

This derivation is perhaps of interest but we shall not make actual use of it. We should mention, though, that the form $\exp \{i\hat{\xi}\Pi\}$ is substantially the form in which this operator appears in the literature; see Wentzel [4, §7].

Second Form of the Operator T

The main advantage of the form $\exp \{(g, A) - (A, g)\}$ of the operator $T\{g\}$ is that its unitary character is immediately apparent. For computation, however, a different form of this operator is preferable.

The form of the operator $T\{g\}$ which we intend to use is

$$T\{g\} = \exp \{-(g, g)/2\} \exp (A, -g) \exp (g, A). \tag{14.28}$$

The operators (g, A) and (A, g) are defined by (14.24). The operator $\exp (g, A)$ which may be defined by its power series

$$\exp (g, A) = \sum_r r\text{¡}(g, A)^r$$

is applicable to every state Φ which has only a finite number of N-components different from zero. Similarly, one defines

$$\exp (A, -g) = \sum_r (-1)^r r\text{¡}(A, g)^r.$$

The advantage of the form (14.28) of $T\{g\}$ is this: Suppose that the M-representation of a state Φ_n is to be found which has only one N-component different from zero. Clearly, the power series for $\exp (g, A)\Phi_n$ breaks off after the $(n + 1)$-st term, since (g, A) involves only the annihilation operator A^-. The power series for $\exp (A, -g)$ can then be applied to each of these $n + 1$ terms.

Before identifying form (14.28) of the operator $T\{g\}$ with form (14.23) we must formulate and prove a few identities which will be important also in later parts of this exposition. These identities are

$$A^+(s') \exp (g, A) = \exp (g, A)B^+(s'), \tag*{(14.29)$^+$}$$

$$A^-(s') \exp (A, -g) = \exp (A, -g)B^-(s'). \tag*{(14.29)$^-$}$$

In order to prove these identities we consider the operators

$$Q^+(\alpha) = \exp (-\alpha g, A)A^+(s') \exp (\alpha g, A) - A^+(s') + \alpha g^+(s'),$$

$$Q^-(\alpha) = \exp (A, \alpha g)A^-(s') \exp (A, -\alpha g) - A^-(s') + \alpha g^-(s'),$$

and differentiate them with respect to α. Using notation (14.27) we obtain

$$\frac{d}{d\alpha} Q^+(\alpha) = -\exp (-\alpha g, A)[(g, A), A^+(s')] \exp (\alpha g, A) + g^+(s'),$$

$$\frac{d}{d\alpha} Q^-(\alpha) = \exp (A, \alpha g)[(A, g), A^-(s')] \exp (A, -\alpha g) + g^-(s').$$

By virtue of

$$[(g, A), A^+(s')] = g^+(s'), \qquad [(A, g), A^-(s')] = -g^-(s')$$

we have

$$\frac{d}{d\alpha} Q^+(\alpha) = 0, \qquad \frac{d}{d\alpha} Q^-(\alpha) = 0$$

whence

$$Q^+(1) = Q^+(0) = 0, \qquad Q^-(1) = Q^-(0) = 0$$

follows and thus $(14.29)^{\pm}$ holds.

From identities $(14.29)^{\pm}$ we immediately verify that the operator $T = T\{g\}$ in the second form satisfies the condition (14.21). An additional effort is needed to show that the operator $T\{g\}$ as defined by (14.28) is unitary. This could be done by identifying the two operators (14.28) and (14.23) but we prefer a direct approach.

We shall establish the identity

$$(14.30) \qquad \exp\,(g_1\,, A) \exp\,(A, g_2) = \exp\,(g_1\,, g_2) \exp\,(A, g_2) \exp\,(g_1\,, A),$$

in which $g_1^+(s)$ and $g_2^-(s)$ are any two functions in $\mathfrak{K}$.

To this end we introduce the function

$$R(\alpha) = \exp\,(\alpha g_1\,, g_2) \exp\,(A, g_2) \exp\,(\alpha g_1\,, A) \exp\,(A, -g_2) \exp\,(-\alpha g_1\,, A)$$

and differentiate it with respect to α. Making use of the relation

$$(g_1\,, A) \exp\,(A, -g_2) = \exp\,(A, -g_2)[(g_1\,, A) - (g_1\,, g_2)],$$

which follows from $(14.29)^-$, we find

$$\frac{d}{d\alpha} R(\alpha) = 0$$

and hence

$$R(1) = R(0) = 1.$$

Thus identity (14.30) is established.[3]

Now we note that the adjoint of the operator $T\{g\}$ as defined by (14.28) is

$$(14.28)^* \qquad T^*\{g\} = T\{-g\} = \exp\,\{-(g, g)/2\} \exp\,(A, g) \exp\,(-g, A).$$

It is then clear that to prove the relations

$$T^*T = 1, \qquad TT^* = 1$$

we need only refer to identity (14.30) for $g_1 = g_2 = -g$, and $g_1 = g_2 = g$ respectively.

Thus it is shown that the operator $T\{g\}$ given by (14.28) is unitary and satisfies condition (14.21).

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[3] The identity of the forms (14.23) and (14.28) of the operator T could be regarded as a special case of a more general identity derived by Feynman [33].

Modified Vacuum State

We are now ready to solve the first specific problem, to find the N-representation of the "modified vacuum state", i.e., the state $\Phi = \Phi_g$ with the M-representation

$$\Phi_g \underset{M}{\longleftrightarrow} \{1, 0, \cdots\}. \tag{14.31}$$

We maintain that this state is given by

$$\Phi_g = T\{-g\}\Phi_0 , \tag{14.32}$$

when Φ_0 is the unmodified vacuum state, represented by

$$\Phi_0 \underset{N}{\longleftrightarrow} \{1, 0, \cdots\}. \tag{14.33}$$

According to the definition of the operator $T = T\{g\}$ the N-representers of the state

$$T\{g\}T\{-g\}\Phi_0$$

are the same as the M-representers of the state $T\{-g\}\Phi_0$. Now, since $T\{g\}T\{-g\}\Phi_0 = \Phi_0$, the N-representers of this state are $\{1, 0, \cdots\}$. The state $T\{-g\}\Phi_0$ has, therefore, just the M-representers that were required by (14.31).

The modified vacuum state Φ_g might also have been defined by the condition

$$B^-(s')\Phi_g = 0. \tag{14.34}$$

From relation (14.21) we then would have

$$A^-(s')T\Phi_g = TB^-(s')\Phi_g = 0;$$

hence there would be a constant c with $|c| = 1$ such that

$$T\Phi_g = c\Phi_0$$

or

$$\Phi_g = cT^{-1}\Phi_0 .$$

The value $c = 1$ for the constant c is, of course, not determined by condition (14.34).

In order to describe explicitly the N-representers of the state $\Phi_g = T\{-g\}\Phi_0$ we use form (14.28). It is clear that the relations

$$\exp(-g, A)\Phi_0 = \sum_r (-1)^r r\text{¡}\left[\int g^+(s)A^-(s)\,dm(s)\right]^r\Phi_0 = \Phi_0 ,$$

and

$$\exp(A, g)\Phi_0 = \sum_n n\text{¡}\left[\int A^+(s)g^-(s)\,dm(s)\right]^n\Phi_0$$

hold. From (14.13) we see that $(A, g)^n\Phi_0$ is a state with only an n-th component, namely

$$\psi_n(s)_n = [n!]^{1/2}g^-(s_1)\cdots g^-(s_n).$$

Hence, the *modified vacuum state* Φ_g has the N-representation

$$(14.35)\qquad \Phi_g \underset{N}{\leftrightarrow} \{[n\text{¡}]^{1/2} g^-(s_1) \cdots g^-(s_n) \exp\{-(g, g)/2\}\}.$$

Note that this function is symmetric.

From this expression we can derive the *probability that a measurement of the number of unmodified particles yields the value n if the field is in the modified vacuum state.* This probability is, obviously,

$$(14.36)\qquad P(N, n \mid M, 0) = n\text{¡}(g, g)^n \exp\{-(g, g)\}.$$

In other words, *the probability that n unmodified particles are in the field if it is void of modified particles is distributed according to Poisson's law.* A corresponding result was found by Bloch and Nordsieck [28], see also Thirring and Touschek [31] and Glauber [34].

The number of unmodified particles expected to be in the state Φ_g is

$$(14.37)\qquad \langle N\rangle_{\Phi_g} = \sum_n nP(N, n \mid M, 0)$$

whence from (14.36),

$$(14.38)\qquad \langle N\rangle_{\Phi_g} = (g, g).$$

This result is in agreement with (13.79).

The M-representation of the unmodified vacuum state is, of course,

$$(14.39)\qquad \Phi_0 \underset{M}{\longleftrightarrow} \{(-1)^m[m\text{¡}]^{1/2} g^-(s_1) \cdots g^-(s_m) \exp\{-(g, g)/2\}\}.$$

The probability of finding m modified particles in this state is

$$(14.40)\qquad P(M, m \mid N, 0) = m\text{¡}(g, g)^m \exp\{-(g, g)\},$$

and the expected value of modified particles in this state is

$$(14.41)\qquad \langle M\rangle_{\Phi_0} = (g, g),$$

in agreement with (13.77).

A further discussion of modified and unmodified vacuum states will be given in Section 15.

Transformation to the Modified Particle Representation

In principle, it is possible to derive the M-representation of any state Φ from the M-representation of the vacuum state Φ_0 if the N-representation of the state Φ is given. For, the state Φ can be obtained from the vacuum state Φ_0 by applying the operator

$$\psi_0 + \int A^+(s_1)\psi_1(s_1)\, dm(s_1) + 2^{-1/2} \iint A^+(s_1)A^+(s_2)\psi_2(s_1, s_2)\, dm(s_1)\, dm(s_2) + \cdots,$$

involving symmetric functions $\psi_n(s)_n$.

We need only replace $A^+(s)$ by $B^+(s) + g^+(s)$, and then apply the resulting operator on the vacuum state Φ_0 given in its M-representation, see (14.39). It is somewhat tedious to carry this out.

Since the operator T is explicitly available, we may derive the result more directly. It is convenient to introduce the functional operator $\mathfrak{T}_{mn}\{g\}$ which transforms the n-th component $\psi_n^-(s)_n$ of the state Φ into this component's contribution to the m-th component $\chi_m^-(s)_m$ of the state $T\{g\}\Phi$; here both components refer to the N-representation. The function $\chi_m^-(s)_m$ is then at the same time the m-th component of the state Φ in the M-representation. We write

$$(14.42) \qquad \chi_m^-(s)_m = T_M^N \psi_n^-(s)_m = \sum_n \mathfrak{T}_{mn}\{g\} \psi_n^-(s)_m$$

using $(s)_m$ instead of $(s)_n$ on the right hand side; for, we would rather indicate the variables of the resulting function than those of the object function. Because of (14.25), (14.26), the inverse transformation $T\{=g\}$ is given by

$$(14.42)^* \qquad \psi_n^-(s)_n = T_N^M \chi_m^-(s)_n = \sum_m \mathfrak{T}_{nm}\{-g\} \chi_m^-(s)_n \, .$$

Before evaluating the expression $\mathfrak{T}_{mn}\{g\}\psi_n^-(s)_m$ explicitly we introduce a few abbreviations. We set

$$(14.43) \qquad g_\rho^\pm(s)_\rho = g^\pm(s_1) \cdots g^\pm(s_\rho), \qquad \rho > 0,$$

$$(14.44) \qquad \psi_n^\pm(s')_\rho(s)_{n-\rho} = \psi_n^\pm(s_1', \cdots, s_\rho', s_{\rho+1}, \cdots, s_{n-\rho}), \qquad \rho > 0,$$

$$(14.45) \qquad g_\rho^\mp \psi_n^\pm(s)_{n-\rho} = \int \cdots \int g_\rho^\mp(s')_\rho \psi_n^\pm(s')_\rho(s)_{n-\rho} \, dm(s_1') \cdots dm(s_\rho'), \qquad \rho > 0.$$

These expressions will also be used for $\rho = 0$; we define

$$(14.43)_0 \qquad g_0^\pm(s)_0 = 1,$$

$$(14.45)_0 \qquad g_0^\mp(s)_0 \psi_n^\pm(s)_n = \psi_n^\pm(s)_n \, .$$

Now let us consider a state Φ_n which has only the n-th N-component $\psi_n(s)_n$ different from zero. The r-th component of the state

$$\exp\,(g,\ A)\Phi_n = \sum_\rho \rho\text{¡}(g,\ A)^\rho \Phi_n$$

is contributed by the term with $\rho = n - r$, since this term involves just $n - r$ annihilations. The r-th component of $\exp\,(g, A)\Phi_n$ is, therefore,

$$[n!r\text{¡}]^{1/2}(n - r)\text{¡}\, g_{n-r}^+ \psi_n^-(s)_r \, .$$

Note that this term is zero unless $r = 0, 1, \cdots, n$; for, $r\text{¡} = 0$ for $r < 0$ and $(n - r)\text{¡} = 0$ for $r > n$.

Now applying the operator

$$\exp\,(A,\ -g) = \sum_\rho \rho\text{¡}(A,\ -g)^\rho$$

to $\exp\,(g, A)\Phi_n$ and multiplying the result by $\exp\,\{-(g, g)/2\}$, we find the

m-th component $\mathfrak{T}_{mn}\{g\}\psi_n(s)_m$ of $T\{g\}\Phi_n$. Evidently, only the term with $\rho = m - r$ contributes since it involves just $m - r$ creations. The result, after summation over r, is

$$\begin{aligned} &\mathfrak{T}_{mn}\{g\}\,\psi_n(s)_m \\ &\qquad = \exp\{-(g, g)/2\} \sum_r (-1)^{m-r}(m!n!)^{1/2} r\text{¡}(n-r)\text{¡}(m-r)\text{¡} \\ &\qquad\qquad \cdot \operatorname{Sy} \overset{-}{g}_{m-r}(s)_{m-r}\overset{+}{g}_{n-r}\overset{-}{\psi}_n(s)_r \; . \end{aligned} \tag{14.46}$$

The symmetrization here refers, of course, to the variables $s_1 , \cdots , s_m$ of which $m - r$ occur in $\overset{-}{g}_{m-r}(s)_{m-r}$ and r in $\overset{+}{g}_{n-r}\overset{-}{\psi}_n(s)_r$. Note that only the terms with $0 \leq r \leq \min(m, n)$ contribute since otherwise $r\text{¡}(n - r)\text{¡}(m - r)\text{¡}$ vanishes.

Replacing g by $-g$ and interchanging n and m we find

$$\begin{aligned} &\mathfrak{T}_{nm}\{-g\}\chi_m(s)_n \\ &\qquad = \exp\{-(g, g)/2\} \sum_r (-1)^{m-r}(m!n!)^{1/2} r\text{¡}(n-r)\text{¡}(m-r)\text{¡} \\ &\qquad\qquad \cdot \operatorname{Sy} \overset{-}{g}_{n-r}(s)_{n-r}\overset{+}{g}_{m-r}\overset{-}{\psi}_m(s)_r \; . \end{aligned} \tag{14.46*}$$

With the expressions (14.46) *we have found explicitly the desired transformation from the N- to the M-representation and its inverse.*

Probabilities in General

We are now ready to compute probabilities. We first compute *the probability*

$$P(M, m \mid N, n)$$

that m modified particles will be found in the field immediately after n unmodified particles have been found. This probability is evidently

$$P(M, m \mid N, n) = \| \mathfrak{T}_{mn}\{g\}\psi_n \|^2 / \| \psi_n \|^2. \tag{14.47}$$

Here

$$\| \psi_n \|^2 = \int | \psi_n(s)_n |^2 \, dm(s)_n , \tag{14.48}$$

and

$$\begin{aligned} \| \mathfrak{T}_{mn}\{g\}\psi_n \|^2 &= \int | \mathfrak{T}_{mn}\{g\}\psi_n(s)_m |^2 \, dm(s)_m \\ &= \exp\{-(g, g)\} m!n! \sum_{r_1} \sum_{r_2} (-1)^{r_1+r_2} \\ &\qquad \cdot r_1\text{¡}r_2\text{¡}(n-r_1)\text{¡}(n-r_2)\text{¡}(m-r_1)\text{¡}(m-r_2)\text{¡} \\ &\quad \cdot \int \overset{+}{g}_{m-r_1}(s)_{m-r_1}\overset{-}{g}_{n-r_1}\overset{+}{\psi}_n(s)_{r_1} \operatorname{Sy} \overset{-}{g}_{m-r_2}(s)_{m-r_2}\overset{+}{g}_{n-r_2}\overset{-}{\psi}_n(s)_{r_2} \, dm(s)_m \; . \end{aligned} \tag{14.49}$$

This formidable expression can be simplified considerably. It can be written in the form

$$\| \mathfrak{T}_{mn}\{g\}\psi_n \|^2 = n! \sum_{\rho} (n-\rho)\text{¡}\,\Theta(m-n,\ \rho;\ (g,\ g))\ \| g_\rho \psi_n \|^2, \tag{14.50}$$

in which we have used the abbreviations

$$\begin{aligned} \| g_\rho \psi_n \|^2 &= \int | g_\rho \psi_n(s)_{n-\rho} |^2 \, dm(s)_{n-\rho} \\ &= \iiint g^{-}_{\rho}(s')_\rho \psi^{+}_{n}(s')_\rho(s)_{n-\rho} g^{+}_{\rho}(s'')_\rho \psi^{-}_{n}(s'')_\rho(s)_{n-\rho} \, dm(s')_\rho \, dm(s'')_\rho \, dm(s)_{n-\rho}, \end{aligned} \tag{14.51}$$

and

$$\Theta(l,\ \rho;\ \kappa) = (2\rho)!(\rho\text{¡})^2 \sum_{\lambda} (-1)^{\rho+\lambda}(\rho+\lambda)\text{¡}(\rho-\lambda)\text{¡}(l+\lambda)\text{¡}\,\kappa^{l+\lambda} \exp\{-\kappa\}. \tag{14.52}$$

Although the summation with respect to λ is to run over all—positive and negative—integer numbers, only the values of λ for which $|\lambda| \leq \rho$ and $\lambda \geq -l$ contribute terms different from zero. Evidently, $\Theta(l,\ \rho;\ \kappa)$ vanishes for $\rho < 0$ and also for $\rho > -l$,

$$\Theta(l,\ \rho;\ \kappa) = 0 \qquad \text{for} \quad \rho < 0 \quad \text{and for} \quad \rho < -l. \tag{14.52}$$

The subscript ρ in (14.50) is, therefore, restricted by

$$n - m \leq \rho \leq n, \tag{14.53}$$

in addition to $\rho \geq 0$.

Derivation

In order to derive formula (14.50) from (14.49) we write the integral occurring on the right side of (14.49) in more explicit forms. First we write it in the form

$$\begin{aligned} &\iiint g^{+}_{m-r_1}(s)_{m-r_1} g^{-}_{n-r_1}(s')_{n-r_1} \psi^{+}_{n}(s')_{n-r_1}(s)_{r_1} \\ &\qquad \cdot \operatorname{Sy} g^{-}_{m-r_2}(s)_{m-r_2} g^{+}_{n-r_2}(s'')_{n-r_2} \psi^{-}_{n}(s'')_{n-r_2}(s)_{r_2} \, dm(s')_{n-r_1} \, dm(s'')_{n-r_2} \, dm(s)_m . \end{aligned}$$

We note that we have written the symmetrization sign Sy before only one of the two factors; the reason is that, automatically, only the symmetric part of the other factor contributes. We next write out all combinations that occur in this symmetrization process. Without giving the details of the combinatorial analysis we state that the triple integral then assumes the form

$$\begin{aligned} &\sum_{r} r\text{¡}(r_1-r)\text{¡}(r_2-r)\text{¡}(m-r_1-r_2+r)\text{¡}\,m\text{¡}\,r_1!r_2!(m-r_1)!(m-r_2)! \\ &\qquad \cdot \left[\int g^{+}(\hat{s})g^{-}(\hat{s})\, dm(\hat{s})\right]^{m-r_1-r_2+r} \int\!\!\int\!\!\int\!\!\int\!\!\int g^{+}_{r_2-r}(s_2)_{r_2-r} g^{+}_{n-r_2}(s'')_{n-r_2} \\ &\qquad \cdot \psi^{+}_{n}(s')_{n-r_1}(s_1)_{r_1-r}(s)_r g^{-}_{r_1-r}(s_1)_{r_1-r} g^{-}_{n-r_1}(s')_{n-r_1} \psi^{-}_{n}(s'')_{n-r_2}(s_2)_{r_2-r}(s)_r \\ &\qquad \cdot dm(s')_{n-r_1} \, dm(s_1)_{r_1-r} \, dm(s'')_{n-r_2} \, dm(s_2)_{r_2-r} \, dm(s)_r . \end{aligned}$$

Here the quintuple integral can be simplified by observing that

$$\iint g^{\pm}_{\sigma-r}(s_0)_{\sigma-r}\, g^{\pm}_{n-\sigma}(\tilde{s})_{n-\sigma}\, \psi^{\mp}_{n}(\tilde{s})_{n-\sigma}(s_0)_{\sigma-r}(s)_r \cdot dm(s_0)_{\sigma-r}\; dm(\tilde{s})_{n-\sigma} = g^{\pm}_{n-r}\psi^{\mp}_{n}(s)_r \;;$$

the last term is defined by (14.45). Consequently, the quintuple integral is independent of r_1 and r_2 and can be written simply as

$$\| g_{n-r}\psi_n \|^2,$$

this term being defined by (14.51).

We now make use of all these simplifications in the right member of (14.49) and cancel various factorials. Setting $n - r = \rho$, $n - r_1 = \rho_1$, $n - r_2 = \rho_2$ we then obtain

$$\| \mathfrak{T}_{mn}\{q\}\psi_n \|^2 = n! \sum_{\rho} \sum_{\rho_1} \sum_{\rho_2} (-1)^{\rho_1+\rho_2}(n-\rho)\text{¡}(\rho-\rho_1)\text{¡}(\rho-\rho_2)\text{¡}$$

$$\cdot \rho_1\text{¡}\rho_2\text{¡}(m - n + \rho_1 + \rho_2 - \rho)\text{¡}(g,\, g)^{m-n+\rho_1+\rho_2-\rho} \exp\{-(g,\, g)\}\; \| g_\rho\psi_n \|^2.$$

The summation with respect to ρ_1 or ρ_2 can be carried out when $\rho_1 + \rho_2$ is kept fixed. Setting $\rho_1 + \rho_2 - \rho = \lambda$ we find

$$\sum_{\rho_1} (\rho - \rho_1)\text{¡}(\rho_1 - \lambda)\text{¡}\rho_1\text{¡}(\rho - \rho_1 + \lambda)\text{¡} = (2\rho)!(\rho\text{¡})^2(\rho + \lambda)\text{¡}(\rho - \lambda)\text{¡}.$$

After insertion of this summation we see that the coefficient of $n!(n-\rho)\text{¡}\, \| g_\rho\psi_n \|^2$ is the same as the expression $\Theta(m - n, \rho; (g, g))$ defined by (14.52). Thus formula (14.50) is proved.

Probabilities in Special Cases

We now compute probabilities in special cases. First we consider the case in which the field is in the vacuum state, $\Phi = \Phi_0$.

After having evaluated the expression $\| \mathfrak{T}_{mn}\{q\}\psi_n \|$ explicitly we can employ formulas (14.47), (14.48), (14.49) to compute the probability $P(M, m \mid N, n)$ that m modified particles are in the field after n unmodified particles have been found.

We first consider the case $n = 0$; i.e. we assume that the field is in an unmodified vacuum. From (14.53) we see that only $\rho = 0$ contributes; from (14.52) we have

$$(14.54) \qquad \Theta(l, 0; \kappa) = l\text{¡}\kappa^l \exp\{-\kappa\},$$

whence, by (14.50) and (14.47),

$$(14.55) \qquad P(M, m \mid N, 0) = m\text{¡}(g, g)^m \exp\{-(g, g)\},$$

in agreement with (14.40).

We add the probability of finding m modified particles in the field after having found one unmodified particle to be present. Setting $n = 1$ we derive from (14.50) the relation

$$\| \mathfrak{T}_{m_1}\{g\}\psi_1 \|^2 = \Theta(m-1, 0; (g, g)) \, \| \psi_1 \|^2 + \Theta(m-1, 1; (g, g)) \, \| g_1\psi_1 \|_2$$

whence, by (14.47),

$$P(M, m \mid N, 1) = \Theta(m-1, 0; (g, g)) + \Theta(m-1, 1; (g, g)) \, \| g_1\psi_1 \|^2 / \| \psi_1 \|^2. \tag{14.56}$$

Here $\Theta(m - 1, 0; (g, g))$ is given by (14.54) while $\Theta(m - 1, 1; (g, g))$ is given by the expression

$$\Theta(l, 1; \kappa) = [(l-1)\rfloor\kappa^{l-1} - 2l\rfloor\kappa^{l} + (l+1)\rfloor\kappa^{l+1}] \exp\{-\kappa\}, \tag{14.57}$$

derived from (14.52). Note that

$$\| g_1\psi_1 \|^2 = \left| \int g^+(s)\psi_1^-(s) \, dm(s) \right|^2 = | g, \psi_1 |^2, \tag{14.58}$$

by (14.51).

We specialize this formula for $m = 0$ and $m = 1$. We have $\Theta(-1, 0; \kappa) = 0$ as seen from (14.54) and (14.57); we find that $\Theta(-1, 1; \kappa) = \exp\{-\kappa\}$. Consequently,

$$P(M, 0 \mid N, 1) = \exp\{-(g, g)\} \, | g, \psi_1 |^2 / \| \psi_1 \|^2. \tag{14.59$_0$}$$

Since $\Theta(0, 0; \kappa) = \exp\{-\kappa\}$ by (14.54) and $\Theta(0, 1; \kappa) = (\kappa - 2)\exp\{-\kappa\}$ by (14.57), we find

$$P(M, 1 \mid N, 1) = \exp\{-(g, g)\}[1 + [(g, g) - 2] \, | g, \psi_1 |^2 / \| \psi_1 \|^2]. \tag{14.59$_1$}$$

We have determined the probability $P(M, m \mid N, n)$ that m modified particles will be found after n unmodified particles have been found. We might just as well ask for *the probability* $P(N, n \mid M, m)$ *that* n *unmodified particles will be found after* m *modified particles have been found.* Since the inverse of the transformation $T\{g\}$ is $T\{-g\}$ it is clear that this probability is found by interchanging n and m and replacing g by $-g$. Consequently,

$$P(N, n \mid M, m) = P(M, m \mid N, n). \tag{14.59}$$

This fact is, of course, a consequence of the general principles of Quantum theory.

Instead of the probabilities $P(N, n \mid M, m) = P(M, m \mid N, n)$ we could calculate various other probabilities, for example, the probability that the field energy H lies between certain values. Probabilities such as this will be calculated, for the modified vacuum state, in Section 17 and again in Section 20.

15. *Transition Probabilities*

At the beginning of Section 14 we formulated the following problem: find the $N(t)$-representers $\psi_n^-(s)_n(t)$ of a state Φ when the $N(0)$-representers $\psi_n^-(s)_n(0)$ are known. We noted, however, that the representers $\psi_n^-(s)_n(t)$ are associated with the creation and annihilation operators

$$A^{\pm}(s, t) = \exp\{\pm it\Omega^S\}[A^{\pm}(s, 0) + q^{\pm}(s) - q^{\pm}(t, s)]; \tag{15.1}$$

here $q^{\pm}(t, s)$ was the abbreviation

$$q^{\pm}(t, s) = \exp\{\mp it\Omega^S\}q^{\pm}(s), \tag{15.2}$$

see (14.4), (13.31), (13.30). We recall that the $N(t)$-representers refer to measurements to be made at the time t; the $N(0)$-representers then refer to measurements made at the time $t = 0$.

We want to show that the $N(t)$-representers can be expressed explicitly in terms of the $N(0)$-representers by the method of canonical transformation explained at the beginning of Section 14.

We observe from (15.1) that the operators

$$\exp\{\mp it\Omega^S\}A^{\pm}(s, t) = A^{\pm}(s, 0) + q^{\pm}(s) - q^{\pm}(t, s) \tag{15.3}$$

are of the type of the operators $B^{\pm}(s)$ treated in the preceding section. For this reason we are able to determine the spectral representers associated with these operators. We shall call these operators "adjusted" creation and annihilation operators and characterize them by a label "ad",

$$A_{ad}^{\pm}(s, t) = \exp\{\mp it\Omega^S\}A^{\pm}(s, t). \tag{15.4}$$

Evidently, these operators satisfy the appropriate commutation rule, and the associated number operator is simply $N(t)$,

$$\int A_{ad}^{+}(s, t)A_{ad}^{-}(s, t)\, dm(s) = N(t). \tag{15.5}$$

The latter fact indicates that the term $N(t)$-representer, which we have used for convenience, is not sufficient to characterize the functions $\psi_n^-(s)_n(t)$. Except for a constant factor,[4] these functions are characterized by the condition

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[4] This constant factor might still depend on t. We determine the functions $\psi_n^-(s)_n(t)$ uniquely by the stipulation that they should be the $N(0)$-representers of the Schrödinger state

$$\Phi_{\ddot{o}}(t) = \exp\{-it\, H_{tot}\}\Phi;$$

here H_{tot} is given by (13.68); see also (15.15). The fact that these $N(0)$-representers may serve as $N(t)$-representers of Φ is implied by the general formalism of quantum theory. In the present case this statement means, according to (15.7), that the $N(0)$-representers $\exp\{-it\, H_{tot}\}$ of $A^{\pm}(s, t)\Phi$ are the $N(0)$-representers of $A^{\pm}(s, 0)\Phi_{\ddot{o}}(t)$. In other words, it means that the relation

$$\exp\{-it\, H_{tot}\}A^{\pm}(s, t)\Phi = A^{\pm}(s, 0)\exp\{-it\, H_{tot}\}\Phi$$

holds. This relation does hold, as seen from formulas (9.1) and (10.9) applied to $B^{\pm}$ instead of $A^{\pm}$.

that the representation

$$\Phi \underset{N}{\overset{t}{\leftrightarrow}} \{\psi_n^-(s)_n(t)\} \tag{15.6}$$

entails the representation

$$\begin{aligned} A^-(s, t)\Phi &\underset{N}{\overset{t}{\leftrightarrow}} \{(n + 1)^{1/2}\psi_{n+1}^-(s)_n(s)(t)\}, \\ A^+(s, t)\Phi &\underset{N}{\overset{t}{\leftrightarrow}} \{n^{1/2}\mathrm{Sy}\ \psi_{n-1}^-(s)_{n-1}(s)(t)\}. \end{aligned} \tag{15.7}$$

If we want to be more specific we shall say, as we have done before, that the representers $\psi_n(s)_n(t)$ are "associated" with the operators $A^{\pm}(s, t)$.

We shall now express the representers associated with the adjusted operators $A_{ad}^{\pm}(s, t)$ in terms of the representers $\psi_n(s, t)$. We introduce the functions

$$\psi_{ad,n}^{\pm}(s)_n(t) = \exp\{\mp it\Sigma^n\Omega^S\}\psi_n^{\pm}(s)(t), \tag{15.8}$$

using the abbreviation

$$\Sigma^n\Omega^S = \Omega^{s_1} + \cdots + \Omega^{s_n}$$

for the sum of the operators Ω^S acting on $\psi_n^{\pm}(s)_n(t)$ considered as a function of $s_1, \cdots, s_n$ respectively, cf. (6.24).

We maintain that the functions $\psi_{ad,n}^-(s)_n(t)$ are "adjusted" $N(t)$-representers of the state Φ associated with the operators $A_{ad}^{\pm}(s, t)$. In fact, from relations (15.6), (15.7), (15.4) we conclude that the representation

$$\Phi \xleftrightarrow[N,ad]{t} \{\psi_{ad,n}^-(s)_n(t)\} \tag{15.9}$$

entails the representation

$$\begin{aligned} A_{ad}^-(s', t)\Phi &\xleftrightarrow[N,ad]{t} \{(n + 1)^{1/2}\psi_{ad,n+1}^-(s)_{n-1}(s')(t)\}, \\ A_{ad}^+(s', t)\Phi &\xleftrightarrow[N,ad]{t} \{n^{1/2}\mathrm{Sy}\ \psi_{ad,n-1}(s)_{n-1}(t)\ \delta(s' - s_n)\}. \end{aligned} \tag{15.10}$$

In order to identify the operators $A_{ad}^{\pm}(s, t)$ with the operators $B^{\pm}(s)$ treated in Section 14, we need only set

$$g^{\pm}(s) = q^{\pm}(s) - q^{\pm}(t, s), \tag{15.11}$$

cf. (13.30). The operator $T\{g\}$ associated with these functions $g^{\pm}(s)$, cf. (14.28), will be denoted by

$$T\{q - q(t)\} = U(t). \tag{15.12}$$

It is evident from the theory developed in Section 14 *that the adjusted $N(t)$-representers of the state are the $N(0)$-representers of the state*

$$\Phi_u(t) = U(t)\Phi. \tag{15.13}$$

We shall term $U(t)$ the *transition operator.*

Clearly, the functional operator $\exp\{\pm it\Sigma^n\Omega^s\}$ is the $\dot{N}(0)$-representer of the operator $\exp\{\pm itH(0)\}$ when $H(0)$ is the unmodified field energy

$$H(0) = \int A^+(s, 0)\Omega^s A^-(s, 0)\, dm(s), \tag{15.14}$$

at the time $t = 0$. The unadjusted $N(t)$-*representers of the state* Φ *are therefore the same functions as the* $N(0)$-*representers* of the state

$$\Phi_s(t) = \exp\{-itH(0)\}U(t)\Phi. \tag{15.15}$$

The time variation of the states $\Phi_w(t)$ or $\Phi_s(t)$ may conveniently be used to describe what happens in the field in the course of time. The state $\Phi_s(t)$ is simply the "Schrödinger" state and its use is said to be a description in the "Schrödinger picture". Use of the state $\Phi_w(t)$ will be called description in the "interaction picture" because it corresponds to Schwinger's "interaction representation".

Clearly, what happens in the field in the course of time is actually described by the probability amplitudes $\psi_n(s)_n(t)$. Whether these amplitudes are considered as representers referring to a varying representation of a fixed state or to a fixed representation of a varying state does not matter, cf. footnote 4.

For our purposes it is preferable to employ the description in the interaction picture. One advantage we shall gain by it stems from the fact that the interaction state $\Phi_w(t)$ becomes constant if the disturbance disappears, while the Schrödinger state $\Phi_s(t)$ would still vary in time.

Transition Probabilities for the Vacuum State

The first question we pose concerns the time variation of the state $\Phi_{0w}(t)$ which results from the *vacuum state* $\Phi_{0w}(0) = \Phi_0$ at the time $t = 0$. From (14.35) we see that this state has the $N(0)$-representation

$$\begin{aligned}\Phi_{0w}(t) \underset{N}{\overset{0}{\longleftrightarrow}} [n!]^{1/2}[(q^-(s_1) - q^-(s_1, t)] \cdots [q^-(s_n) - q^-(s_n)(t)])\\ \cdot \exp\{-(q - q(t), q - q(t))/2\}.\end{aligned} \tag{15.16}$$

Using the momentum k as the quantum variable s we may write this representation in the form

$$\begin{aligned}\Phi_{0w}(t) \underset{N}{\overset{0}{\longleftrightarrow}} \{[n!]^{1/2}q_n^-(k)_n(1 - \exp\{it\omega_1\}) \cdots (1 - \exp\{it\omega_n\})\\ \cdot \exp\{\mathcal{R}e\,[q(t) - q]\}\}.\end{aligned} \tag{15.17}$$

Here

$$\omega = \omega(k) = [\mu^2 + |k|^2]^{1/2}$$

and

$$(15.18)\qquad (q, q(t) - q) = \int |q(k)|^2 \exp\{it\omega - 1\}\, dk.$$

The probability that n particles will be found at the time t if the field was in the vacuum state at the time $t = 0$ is then

$$(15.19)\quad P(N(t), n \mid N(0), 0) = n\text{¡}[2\, \mathcal{R}e\, (q(t) - q, q)]^n \exp\{2\, \mathcal{R}e\, (q(t) - q, q)\},$$

see (14.36). The probability thus calculated is also called a *transition probability*.

The expected value of the number of particles,

$$(15.20)\qquad \langle N(t)\rangle_{\Phi_0} = \langle N(0)\rangle_{\Phi_{0w}} = 2\, \mathcal{R}e\, (q - q(t), q),$$

could already have been determined in Section 13, see (13.41).

It is interesting to investigate what happens when the time t increases indefinitely. One may ask in particular whether or not the representation (15.16) of the state $\Phi_{0w}(t)$ has a limit as $t \rightarrow \infty$. It is seen from formula (15.17) that this is not the case. Consequently, *no scattering operator* in the strict sense *exists* for the process under consideration.

The scattering operator, introduced by Heisenberg [29], can be defined as the unitary operator which transforms the limit $\Phi_w(-\infty)$ of the interaction state $\Phi_w(t)$ for $t \rightarrow -\infty$ into the limit $\Phi_w(\infty)$ of $\Phi_w(t)$ for $t \rightarrow \infty$.

One might also try to characterize the scattering operator as the limit of the "incomplete" scattering operator which transforms the state $\Phi_w(-t)$ into the state $\Phi_w(t)$. This transformation is evidently the same as that of $\Phi_w(0)$ into $\Phi_w(2t)$, since the time does not enter the problem explicitly. Therefore, the incomplete scattering operator in our case is, simply, the operator $U(2t)$. The fact that $\Phi_{0w}(2t) = U(2t)\Phi_0$ has no limit as $t \rightarrow \infty$ implies, of course, that the operator $U(2t)$ has no limit; in other words, that no scattering operator exists.

Nevertheless, definite statements can be made about the *asymptotic value of the transition probabilities*. We first assume $(q, q) < \infty$. As in Section 13 we also assume that the field extends over the infinite space. From Lemma 1, see (13.45), we may then deduce the relation

$$(15.21)\qquad (q(t), q) \rightarrow 0 \qquad \text{as} \qquad t \rightarrow \infty.$$

Consequently, we may conclude from (15.19) that the probability $P(N(t), n \mid N(0), 0)$ has a limit as $t \rightarrow \infty$. Denoting this limit by $P(N_\infty, n \mid N(0), 0)$ we have

$$(15.22)\qquad P(N_\infty, n \mid N(0), 0) = n\text{¡}2^n(q, q)^n \exp\{-2(q, q)\}.$$

The distribution of this *asymptotic transition probability* again obeys Poisson's law, just as $P(M, m \mid N, 0)$ does, but the expected value of the number N_∞ is

$$(15.23)\qquad \langle N\rangle_{\Phi_0} = 2(q, q),$$

in agreement with (13.53), and not (q, q) as it is for the probabilities $P(M, m \mid N, 0)$.

If $(q, q) = \infty$, one easily derives the relations

$$(q(t) - q, q) \to \infty, \quad \text{as} \quad t \to \infty, \tag{15.21$_\infty$}$$

$$P(N(t), n \mid N(0), 0) \to 0 \quad \text{as} \quad t \to \infty, \quad \text{and} \tag{15.22$_\infty$}$$

$$\langle N(t) \rangle_\Phi \to \infty \quad \text{as} \quad t \to \infty. \tag{15.23$_\infty$}$$

These relations express the "infrared catastrophe": the probability that the field consists of a finite number of particles approaches zero, while the expected number of particles increases indefinitely.

General Transition Probabilities

So far, we have been concerned with the case in which the field was in a state of vacuum initially. We proceed to describe the *transition operator* $U(t)$ and the *transition probabilities* $P(N(t), n \mid N(0), n_0)$ quite generally. This can easily be done on the basis of results obtained in Section 14. From relation (14.30) combined with (15.11) we obtain the expression

$$\begin{aligned} U(t) &= \exp \{\mathcal{R}e\,(q(t) - q, q)\} \\ &\quad \cdot \exp (A(0), q(t) - q) \exp (q - q(t), A(0)) \end{aligned} \tag{15.24}$$

for the transition operator. Metaphorically speaking, we may say that the transition operator $U(t)$ effects first a series of annihilations described by $\exp (q - q(t), A(0))$ and then a series of creations described by $\exp (A^+(0), q(t) - q)$.

The term which the n_0-th component of the state contributes to the n-th component of the state $\Phi_\infty(t) = U(t)\Phi$ could be computed explicitly by inserting $g = q - q(t)$ into formula (14.46). We shall not carry this out in detail.

By insertion into formulas (14.47), and (14.50) we find the transition probability

$$\begin{aligned} &P(N(t), n \mid N(0), n_0) \\ &\quad = n_0! \sum_\rho (n_0 - \rho)!\, \Theta(n - n_0\,, \rho; 2\,\mathcal{R}e\,(q - q(t), q)) \\ &\quad \cdot \|\,(q - q(t))_\rho \psi_n\,\|^2 / \|\,\psi_n\,\|^2. \end{aligned} \tag{15.25}$$

Again we shall give no details. We confine ourselves to determining the asymptotic value of the transition operator and the transition probability in the case $(q, q) < \infty$.

First we mention the relation

$$(q(t), A(0))\Phi \to 0 \quad \text{as} \quad t \to 0. \tag{15.26}$$

To prove this relation we start with the $N(0)$-representation

$$(q(t),\ A)\Phi \overset{0}{\underset{N}{\leftrightarrow}} \{(n+1)^{1/2} \int q^{+}(s)\psi^{-}_{n+1}(s)_n(s)\, dm(s)\}$$

of the state $(q(t),\ A(0))$, in which $\psi_n(s)_n$ denotes the $N(0)$-representer of the state Φ. Each component on the right side approaches zero, as follows from Lemma 1 of Section 13, see (13.45). The statement $\|\,(q(t),\ A)\Phi\,\| \rightarrow 0$ then follows by arguments similar to those used in the proof of Lemma 2.

As a consequence of relation (15.26) we have

$$\exp\,(q - q(t),\ A(0)) \rightarrow \exp\,(q,\ A(0)) \quad \text{as} \quad t \rightarrow \infty. \tag{15.27}$$

Thus the annihilation factor of the transition operator $U(t)$ approaches a definite limit.

The same is not the case, however, for the creation part of the operator $U(t)$, since no analogue to relation (15.26) holds for $(A(0),\ q(t))\Phi$. This remark confirms the statement made before that *the transition operator $U(t)$ has no limit as $t \rightarrow \infty$.*

The transition probabilities $P(N(t),\ n \mid N(0),\ n_0)$, on the other hand, *have a limit $P(N_\infty,\ n \mid N(0),\ n_0)$*. In order to deduce this limit from formula (15.25) we must determine the limit of the term $\|\,(q - q(t))_\rho\psi_n\,\|^2$. To this end we insert $g = q - q(t)$ in expression (14.51) of this term and then apply Lemma 1. It follows from this lemma that

$$g^{\mp}_{\rho}\psi^{\mp}_{n}(s)_{n-\rho} = \int \cdots \int g^{\mp}(s_1') \cdots g^{\mp}(s_\rho')\psi^{\pm}_{n}(s_1',\ \cdots,\ s_\rho')(s)_{n-\rho}\, dm(s_1') \cdots dm(s_\rho')$$

approaches the function

$$\begin{aligned} &q^{\mp}_{\rho}\psi^{\pm}_{n}(s)_{n-\rho} \\ &\quad = \int \cdots \int q^{\mp}(s_1') \cdots q^{\mp}(s_\rho')\psi^{\pm}_{n}(s_1',\ \cdots,\ s_\rho')(s)_{n-\rho}\, dm(s_1') \cdots dm(s_\rho'). \end{aligned} \tag{15.28}$$

Hence

$$\|\,(q - q(t))_\rho\psi_{n_0}\,\|^2 \quad \text{as} \quad t \rightarrow \infty. \tag{15.29}$$

Insertion in (15.25) yields the expression

$$\begin{aligned} &P(N_\infty,\ n \mid N(0),\ n_0) \\ &\quad = n_0!\sum_{\rho} (n_0 - \rho)\text{¡}\Theta(n - n_0,\ \rho;\ 2(q,\ q))\ \|\,q_\rho\psi_{n_0}\,\|^2/\|\,\psi_{n_0}\,\|^2. \end{aligned} \tag{15.30}$$

This is the final formula for *the asymptotic transition probability*. It is remarkable that this probability depends on the numbers n, n_0 and on the function q essentially as the probability $P(M,\ m \mid N,\ n)$ depends on m, n, q; the only difference is that $2(q,\ q)$ takes the place of $(q,\ q)$.

Note that only the $n_0 + 1$ terms with $\rho = 0, \cdots, n + 1$ contribute to the series (15.30).

The term in this series for which $\rho = 0$ is

$$P_{sp}(N_\infty, n \mid N(0), n_0) = (n - n_0)\text{¡}2^{n-n_0}(q, q)^{n-n_0} \exp\{-2(q, q)\}, \tag{15.31}$$

since

$$\Theta(l, 0; \kappa) = l\text{¡}\kappa^l \exp\{-\kappa\}$$

by (14.54). We may also write

$$\begin{aligned} &P_{sp}(N_0, n \mid N(0), n_0) \\ &\quad = [P(N_\infty, n_0 \mid N(0), 0)]^{-1}P(N_\infty, n \mid N(0), 0) \exp\{-2(q, q)\}. \end{aligned} \tag{15.32}$$

This contribution does not depend on the function $\psi_{n_0}(s)_{n_0}$, i.e., on the state of the n_0 particles initially present; it is, therefore, called the probability of *spontaneous emission.* Note that this term vanishes if $n < n_0$.

The terms in the series (15.30) for $P(N_\infty, n \mid N(0), n_0)$ for which $\rho > 0$ are said to refer to *induced emission* and *absorption,* if $n > n_0$ and $n < n_0$, respectively, or to *balanced emission and absorption,* if $n = n_0$.

Suppose the state of the n_0 particles initially present is such that the function $\psi_{n_0}(s)_{n_0}$ which described this state satisfies the relations

$$g_\rho \psi_{n_0}(s)_{n_0-\rho} = 0 \qquad \text{for} \qquad \rho = 1, 2, \cdots, n_0;$$

then no induced effects occur. For $n_0 = 1$ these relations reduce to the orthogonality relation

$$\int g^+(s)\psi^-(s)\, dm(s) = 0.$$

This is therefore the condition that no emission is induced when a single particle is present.

We now proceed to derive from (15.30) explicitly the values of the *transition probabilities for special values of n_0 and n.*

For $n_0 = 0$ we find again formula (15.22). For $n_0 = 1$ we find

$$\begin{aligned} P(N_\infty, n \mid N(0), 1) &= \Theta(n - 1, 0; 2(q, q)) \\ &\quad + \Theta(n - 1, 1; 2(q, q)) \,\|\, q_1\psi_1 \,\|^2/\|\, \psi_1 \,\|^2. \end{aligned} \tag{15.33}$$

Here $\Theta(n - 1, 0; 2(q, q))$ and $\Theta(n - 1, 1; 2(q, q))$ are given by (14.54) and (14.57). The term $\|\, q_1\psi_1 \,\|$ is defined by

$$\|\, q_1\psi_1 \,\|^2 = \left| \int q^+(s)\psi_1^-(s)\, dm(s) \right|^2 = |\, q, \psi_1 \,|^2,$$

cf. (14.60). Accordingly,

$$\begin{aligned} P(N_\infty, n \mid N(0), 1) &= (n-1)¡\,2^{n-1}(q, q)^{n-1} \exp\{-2(q, q)\} \\ &\quad + 2^{n-2}(q, q)^{n-2}[(n-2)¡ - 4(n-1)¡(q, q) + 4n¡(q, q)^2] \\ &\quad \cdot \exp\{-2(q, q)\} \mid q, \psi_1 \mid^2 / \| \psi_1 \|^2. \end{aligned} \tag{15.34}$$

The first term of the right member of (15.32) is obviously independent of ψ_1; hence it is independent of the state of the single particle initially present. This term is therefore the probability of the spontaneous emission of n particles. The second term in the right member of (15.34) is accordingly the probability of induced emission of n particles. We note that no induced effects occur if the representer $\psi_1(s)$ of the particle present initially is orthogonal to the function $q(s)$.

The term emission used here is appropriate only for $n > 1$. If $n = 0$, we should speak of absorption. There is no spontaneous absorption, since $(-1)¡ = 0$. For $n = 0$ we find the *probability of the induced absorption of one particle if one particle was present initially*

$$P(N_\infty, 0 \mid N(0), 1) = \exp\{-2(q, q)\} \mid q, \psi_1 \mid^2 / \| \psi_1 \|^2. \tag{15.34$_0$}$$

For $n = 1$ we find *the probability of balanced emission and absorption*, i.e. the probability that the number of particles does not change, *if one particle was present initially*

$$\begin{aligned} P(N_\infty, 1 \mid N(0), 1) &= \exp\{-2(q, q)\} \\ &\quad + 2[(q, q) - 1] \exp\{-2(q, q)\} \mid q, \psi_1 \mid^2 / \| \psi_1 \|^2. \end{aligned} \tag{15.34$_1$}$$

Finally, we write out explicitly *the probability of the emission of one particle, if one particle was present initially*, setting $n = 2$,

$$\begin{aligned} P(N_\infty, 2 \mid N(0), 1) &= 2(q, q) \exp\{-2(q, q)\} \\ &\quad + [2(q, q)^2 - 4(q, q) + 1] \exp\{-2(q, q)\} \mid q, \psi_1 \mid^2 / \| \psi_1 \|^2. \end{aligned} \tag{15.34$_2$}$$

Perturbation

The general expression (15.30) for $P(N_\infty, n \mid N(0), n_0)$ is somewhat involved. For many purposes one might be satisfied with an *approximation* to this expression valid if the disturbance is sufficiently small. We substitute ϵq for q, expand with respect to powers of ϵ up to the second order, and then set $\epsilon = 1$.

From (14.54) we find

$$\begin{aligned} &\Theta(0, 0; \kappa) = 1 - \kappa, &&\Theta(0, 1; \kappa) = -2, \\ &\Theta(1, 0; \kappa) = \kappa, &&\Theta(1, 1; \kappa) = 1, \\ & &&\Theta(-1, 1; \kappa) = 1, \end{aligned} \tag{15.35}$$

within terms of second order, while $\Theta(l, \rho; \kappa)$ vanishes of at least second order for all other values of l and ρ.

Accordingly, we obtain from (15.30) the approximate expressions

$$
\begin{aligned}
&P(N_\infty, n_0 - 1 \mid N(0), n_0) = n_0 \, \| q_1\psi_{n_0} \|^2 / \| \psi_{n_0} \|^2, \\
(15.36) \qquad &P(N_\infty, n_0 \mid N(0), n_0) \quad = 1 - 2(q, q) - 2n_0 \, \| q_1\psi_{n_0} \| / \| \psi_{n_0} \|^2, \\
&P(N_\infty, n_0 + 1 \mid N(0), n_0) = 2(q, q) + n_0 \, \| q_1\psi_{n_0} \|^2 / \| \psi_{n_0} \|^2,
\end{aligned}
$$

within terms of third order in q, while all other transition probabilities vanish to at least third order in q. Here the term $\| q_1\psi_{n_0} \|^2$ is given in detail by

$$
(15.37) \qquad \| q_1\psi_{n_0} \|^2 = \int \left| \int q^+(s')\psi_{n_0}^-(s)_{n_0-1}(s') \, dm(s') \right|^2 dm(s)_{n_0-1}
$$

according to (14.53).

It is remarkable that *in first order the contributions to the transition probability due to induced effects are proportional to the number of particles present initially*, while the contribution due to spontaneous emission is independent of this number.

We note that no induced effects take place if the state of the n particles present initially is such that

$$
q_1\psi_{n_0}(s)_{n_0-1} = \int q^+(s')\psi_{n_0}^-(s)_{n_0-1}(s') \, dm(s') = 0.
$$

At the same time we note that the term $q_1\psi_{n_0}(s)_{n_0-1}$ may be a function with a finite norm $\| q_1\psi_n \|$ for appropriate functions $\psi_{n_0}(s)_{n_0}$ even in cases in which $\| q \| = \infty$. Under these circumstances one could apparently calculate the transition probability of induced emission and absorption in the lowest order; actually every probability $P(N_\infty, n \mid N(0), n)$ is zero in this case.

The formulas which we have derived from an expansion of the complete expression for the transition probabilities could also have been derived by a straightforward *perturbation procedure.* One need only assume an expansion of the representers $\psi_n(s, t)$ in powers of ϵ and determine the coefficients successively. We do not intend to give details.

Method of Spectral Transformation

In conclusion we want to mention *another approach to determining the transition operator* $U(t)$ and the transition probabilities, namely, the approach in which the spectral representation of the total energy H_{tot} is employed. Since the M-representation yields such a spectral representation we may describe this approach as follows: Suppose the state Φ is described in the $N(0)$-representation in terms of the representers $\psi_n(s)_n$. Then we may first find its $M(0)$-representers by employing the transformation $T\{q\}$ as described in Section 14. The time variation of the M-representers is immediately found from the fact that the total energy H_{tot} is constant. Accordingly, the $M(t)$-representers are obtained by applying the operator $\exp\{-itH\}$. We can now obtain the $N(t)$-representation of Φ, or the $N(0)$-representation of $\Phi_\delta(t)$, by applying the inverse transformation $T\{-q\}$. Comparison of the resulting expression with expression (15.15) yields the identity

$$U(t) = \exp\{itH\}T\{-q\}\exp\{-itH\}T\{q\}. \tag{15.39}$$

Clearly, expression (15.12) for $U(t)$ is much more concise than expression (15.39).

We have described this approach because the method of "spectral transformation of the energy" is of general importance; moreover, we can draw from it an interesting conclusion concerning the asymptotic transition probabilities.

Suppose n_0 particles are initially present in the state Φ_{n_0}. We write the m-th component $\chi_m(s)_m$ of the m-representation of this state as

$$\chi_m(s)_m = \mathfrak{T}_{mn_0}\{q\}\,\psi_{n_0}(s)_m\,,$$

using the notation (14.44). The n-th component of the $N(t)$-component is then

$$\psi_n(s)_n(t) = \mathfrak{T}_{nm}\{-q\}\,\exp\{-it\,\Sigma^m\,\Omega^S\}\chi_m(s)_n\,,$$

by virtue of (15.38). The probability $P(N(t),\, n \mid N_0\,,\, n_0)$ can therefore be written as

$$P(N(t),\, n \mid N(0),\, n_0) = \|\,\Sigma_m\,\mathfrak{T}_{nm}\{-q\}\,\exp\{-it\,\Sigma^m\,\Omega^S\}\mathfrak{T}_{mn_0}\{q\}\,\psi_{n_0}\,\|^2.$$

We now consider the limit of this expression as the time t increases indefinitely. We maintain that the cross terms in the square of the sum approach zero as the time t increases indefinitely. This fact can be derived by arguments in which Lemma 1, see (13.45), is used; we shall not give details. The result can be simply stated as the identity

$$P(N_\infty\,,\, n \mid N(0),\, n_0) = \sum_m P(N_\infty\,,\, n \mid M,\, m)P(M,\, m \mid N(0),\, n_0). \tag{15.40}$$

This formula agrees with *the classical rule for the composition of probabilities.* This rule holds, therefore, even in the case in which the number of modified particles is not measured after the initial measurement and before the final measurement of the unmodified number of particles.

16. *Boson Fields under the Influence of a Source Distribution Which Varies in Time*

In the preceding section the source density $j(x)$ was assumed constant in time. We now want to show that the formulas developed in Section 14 can also be used to describe the transitions in cases in which the source density $j = j(x, t)$ varies in time. Appropriate conditions to be imposed on the behavior of the function $j(x, t)$ will be formulated below. These conditions concern, in particular, the behavior of $j(x, t)$ as $t \to \pm\infty$.

In accordance with previous formulations, the creation and annihilation operators $A^{\pm}(s, t)$ should satisfy the differential equation

$$(16.1) \qquad \nabla_t A^{\pm}(s, t) \mp i\Omega^S A^{\pm}(s, t) = \mp i\Omega^S q^{\pm}(s, t),$$

in which the function $q^{\pm}(s, t)$ is given by

$$(16.2) \qquad q^{\pm}(s, t) = 2(2\Omega^S)^{-3/2} j^{\pm}(s, t),$$

cf. (13.18). Solutions of these differential equations are sought which agree with given annihilation and creation operators as $t \to -\infty$.

We shall not prescribe the operators $A^{\pm}(s, t)$ for $t = -\infty$ directly; in fact, that would be impossible. Instead, we prescribe operators $A^{\pm}_0(s)$ which would be the operators $A^{\pm}(s, 0)$ if no sources were present up to the time $t = 0$. The operators $A^{\pm}_0(s)$ are prescribed by implication, namely by giving for any state Φ the N_0-representation associated with the operators $A^{\pm}_0(s)$,

$$(16.3) \qquad \Phi \underset{N_0}{\leftrightarrow} \{\psi_n(s)_n\}.$$

If no sources had been present up to the time t, the operators $A^{\pm}(s, t)$ would have been given by

$$\exp\{\pm it\Omega^S\} A^{\pm}_0(s).$$

Implying that the effect of the sources vanishes as $t \to -\infty$, we write the initial conditions imposed on the operators $A^{\pm}(s, t)$ in the form

$$(16.4) \qquad \exp\{\mp it\Omega^S\} A^{\pm}(s, t) \to A^{\pm}_0(s) \quad \text{as} \quad t \to -\infty.$$

Note that the left members here are the "adjusted" operators, cf. (15.4). With the aid of

$$(16.5) \qquad g^{\pm}(s, t) = \pm i\Omega^S \int_{-\infty}^{t} \exp\{\mp it'\Omega^S\} q^{\pm}(s, t')\, dt',$$

the solutions of equations (16.1) which satisfy condition (16.4) are easily described. These solutions are

$$(16.6) \qquad A^{\pm}(s, t) = \exp\{\pm it\Omega^S\}[A^{\pm}_0(s) - g^{\pm}(s, t)],$$

as is immediately verified.

The conditions on the function $j(x, t)$ referred to above will now be formulated: The function $j(x, t)$ should behave in such a way that the integration entering the definition of $g^{\pm}(s, t)$ can be performed, that the resulting function $g^{\pm}(s, t)$ belongs to $\mathfrak{K}$, i.e. that the norm $\| g(t) \|$ is finite, and that

$$(16.7) \qquad \| g(t) \| \to 0 \quad \text{as} \quad t \to -\infty.$$

If the function $j(x, t)$ satisfies these requirements we say *the source distribution has been switched on properly.* Note that we do not require the functions $j(s, t)$ or $q(s, t)$ to belong to $\mathfrak{K}$.

If the function $j(x, t)$ is specialized to be zero for $t < 0$ and to be constant in time for $t > 0$, relation (16.5) reduces, of course, to relation (15.11). In fact, the problem treated in Section 15 may be included among the problems treated here: it results when no sources are present up to the time $t = 0$ and constant sources are switched on instantaneously at the time $t = 0$. The operators $A^{\pm}(s, 0)$ of Section 15 then agree with the operator $A_0^{\pm}(s)$ of the present section.

Relation (16.6) is of the same type as relation (15.1); therefore the transformation of a representation associated with the operators $A_0^{\pm}(s)$ into a representation associated with the operators $A^{\pm}(s, t)$ can be given explicitly.

As in Section 15, we find that the transition can be more easily described by the Schrödinger picture or the interaction picture than by the Heisenberg picture. We denote here as Schrödinger state the state $\Phi_s(t)$ whose N_0- representers are the $N(t)$-representers of the state Φ. The interaction state $\Phi_w(t)$ is connected with the Schrödinger state by relation

$$\Phi_s(t) = \exp\{-itH_0\}\Phi_w(t), \tag{16.8}$$

in which

$$H_0 = \int A_0^+(s)\Omega^S A_0^-(s)\, dm(s).$$

From the arguments at the beginning of Section 15 we conclude that the interaction state is given by

$$\Phi_w(t) = U(t)\Phi = T_0\{g(t)\}\Phi; \tag{16.9}$$

the subscript $_0$ indicates that the operators $A_0^{\pm}$ are to take the place of $A^{\pm}$ in (14.23) or (14.28). Note that relation (16.7) implies that, as $t \to -\infty$, the transition operator $U(t) = T_0\{g(t)\}$ approaches the identity,

$$T_0\{g(t)\} \to 1 \quad \text{as} \quad t \to -\infty. \tag{16.10}$$

In order to discuss the implications of formula (16.9) we consider the problem of *switching on a constant source distribution.* In other words, we assume

$$j(x, t) = j(x) \quad \text{for} \quad t^- < t < \infty, \tag{16.11}$$

but we do not specify what the function $j(x, t)$ is for $t < t^-$. The function $g^{\pm}(s, t)$ given by (16.5) may, in this case, be written in the form

$$g^{\pm}(s, t) = g^{\pm}(s, t^-) + \hat{q}^{\pm}(t^-, s) - \hat{q}^{\pm}(t, s) \quad \text{for} \quad t > t^-. \tag{16.12}$$

Here $\hat{q}^{\pm}(t, s)$ is defined by

$$\hat{q}^{\pm}(t, s) = \exp\{\mp it\Omega^S\}q^{\pm}(s) \tag{16.13}$$

and $q^{\pm}(s)$ is derived from $j(x)$ in accordance with (16.2). We are using the "caret" although we did not use it for the same function in Section 15; we want to emphasize that the function $\hat{q}^{\pm}(t, s)$ should not be confused with the function $q^{\pm}(s, t)$ associated with the source distribution $j(x, t)$.

The term $g^{\pm}(s, t^-)$ in formula (16.12) embodies the effect of the process of switching on the source distribution. The other terms are obviously independent of this process. If the source is switched on instantaneously at the time $t = t^-$ the first term vanishes, $g^{\pm}(s, t^-) = 0$. This agrees with formula (15.11) if one sets $t^- = 0$.

Infinitely Slow Switch-on

We say a source distribution is switched on *infinitely slowly*,[5] "quasistatically", or "adiabatically", up to the time $t = t^-$ if

$$(16.14) \qquad g^{\pm}(s, t) = -\hat{q}^{\pm}(t, s, t^-) = -\exp\{\mp it\Omega^S\}q^{\pm}(s, t), \qquad t \leq t^-.$$

To justify this definition let us first consider a source distribution which has been switched on linearly during a finite time interval,

$$(16.15) \qquad j(x, t) = \begin{cases} 0, & -\infty < t \leq t_0, \\ (t^- - t_0)^{-1}(t - t_0)j^{\pm}(x, t^-), & t_0 \leq t \leq t^-. \end{cases}$$

From (16.5) we find

$$g^{\pm}(s, t) = -[(t^- - t_0)^{-1}(t - t_0) - [\mp i(t^- - t_0)\Omega^S]^{-1}]\hat{q}^{\pm}(t, s, t^-)$$
$$- [\mp i(t^- - t_0)\Omega^S]^{-1}\hat{q}^{\pm}(t_0, s, t^-) \qquad \text{for} \qquad t_0 \leq t \leq t^-.$$

Letting t_0 approach $-\infty$ we find

$$(16.16) \qquad g^{\pm}(s, t) \to -\hat{q}^{\pm}(t, s, t^-) \qquad \text{for} \qquad t \leq t^-;$$

hence, in particular, $g^{\pm}(s, t^-) \to -\hat{q}^{\pm}(t, s, t^-)$.

It is important to realize that condition (16.14) for an infinitely slow process of switching on implies that the function $q^{\pm}(s, t^-)$ belongs to $\mathfrak{K}$, i.e. $(q(t^-), q(t^-)) < \infty$. On the other hand, it is not necessary to impose this condition for the process of linear switching on, provided $(q, \Omega q) < \infty$. The breakdown of the limit process if $(q(t^-), q(t^-)) = (q, q) = \infty$ is apparent when one realizes that the transition operator $T_0\{-g(t^-)\}$, which is still defined for the linear switch on, does not converge to a transition operator. We express this situation by saying that *it is impossible to switch on a source distribution infinitely slowly up to the time* $t = t^-$ *if* $(q(t^-), q(t^-)) = \infty$. We may also say that it is impossible to create an infrared catastrophe by switching on the sources infinitely slowly as required by (16.7).

The process of infinitely slow switch on can also be used to handle source distributions $j(x, t)$ which as $t \to -\infty$ do not die out. To this end, the process described by the function $j(x, t)$ may be replaced by an infinitely slow process in which the source $j(x, t^-)$ is attained at the time $t = t^-$. To this modified process, according to (16.14), (16.5), the function

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[5] For such processes in connection with single particles see [27], [32].

$$g^{\pm}(t, s) = -\exp\{\mp it^{-}\Omega^{S}\}q^{\pm}(s, t^{-})$$

(16.17)

$$\pm\, i\Omega^{S}\int_{t^{-}}^{t} \exp\{\mp it'\Omega^{S}\}q^{\pm}(s, t')\, dt'$$

is assigned. Assuming the function $q^{\pm}(s, t)$ to possess a continuous derivative with respect to t, we may write formula (16.17) in the form

$$g^{\pm}(t, s) = -\exp\{\mp it\Omega^{S}\}q^{\pm}(s, t) + \int_{t^{-}}^{t} \exp\{\mp it'\Omega^{S}\}\, dq^{\pm}(s, t').$$

We now let t^{-} approach $-\infty$. We assume that the limit function

$$g^{\pm}(t, s) = -\exp\{\mp it\Omega^{S}\}q^{\pm}(s, t) + \int_{-\infty}^{t} \exp\{\mp it'\Omega^{S}\}\, dq^{\pm}(s, t') \tag{16.18}$$

exists and satisfies condition (16.7) and assign this function to the switching on process described by the function $j(x, t)$. Formula (16.18) holds, of course, also if the source distribution $j(x, t)$ approaches zero sufficiently strongly as $t \to -\infty$.

We see from formulas (16.18) and (16.14) that a process of infinitely slow switch on up to the time $t = t^{-}$ can also be characterized by the condition

$$\int_{-\infty}^{t^{-}} \exp\{\mp it'\Omega^{S}\}\, dq^{\pm}(s, t') = 0, \tag{16.19}$$

provided a continuous derivative $dq^{\pm}(s, t)/dt$ exists. This relation can not be satisfied for all values of s unless $dq\,(s, t) = 0$ for $t < t^{-}$; i.e. if $q(s)$ is time-independent. We assume here that $q(s, t)$ satisfies a suitable condition such as

$$\int_{-\infty}^{t^{-}} |\, dq(s, t)/dt \,|\, dt < \infty. \tag{16.20}$$

Since a time-independent source distribution does not satisfy condition (16.7) we may say: *An infinitely slow process of switching on cannot be realized as a proper switching on process; it is always to be considered the limit of such proper processes.*

Suppose the source distribution is *constant* after it has been switched on infinitely slowly. Then we have

$$g^{\pm}(s, t) = -\hat{q}^{\pm}(t, s) \qquad \text{for} \qquad t \geq t^{-}; \tag{16.21}$$

thus the process remains "infinitely slow" from $t = t^{-}$ on. The operators $A^{\pm}(s, t)$ are then given by

$$A^{\pm}(s, t) = \exp\{\pm it\Omega^{S}\}A_{0}^{\pm}(s) + q^{\pm}(s) \qquad \text{for} \qquad t > t^{-}, \tag{16.22}$$

as seen from (16.6). This relation may also be expressed in the form

$$B^{\pm}(s, t) = \exp\{\pm it\Omega^{S}\}A_{0}^{\pm}(s, t) \qquad \text{for} \qquad t > t^{-}, \tag{16.23}$$

which involves the modified creation and annihilation operators introduced in Section 14. This formula has an interesting consequence: Suppose the field is originally in the vacuum state Φ_0, so that $A^-_0(s)\Phi_0 = 0$; then $B^-(s, t)\Phi_0 = 0$ for $t \geq t^-$. In other words, the state is in the modified vacuum state from $t = t^-$ on. Similarly we find that an eigenstate of the number operator N_0 goes over into a corresponding eigenstate of the modified number operator M. Thus *the stationary eigenstates of the modified number can be produced from the corresponding eigenstates of the unmodified number by switching on the source distribution infinitely slowly.*

Removing Sinks from Sources

The following particular switching on process is of considerable interest: Two congruent source and sink distributions are originally superimposed so that they cancel each other; the source distribution is then kept stationary while the sink distribution is removed to infinity. Describing the path of the moved sinks by a continuous function $a(t)$ for which

$$(16.24)\qquad \begin{aligned} a(t) &\rightarrow 0 &\quad \text{as} \quad & t \rightarrow -\infty, \\ |\, a(t)\, | &\rightarrow \infty &\quad \text{as} \quad & t \rightarrow 0, \end{aligned}$$

we may express this process by

$$(16.25)\qquad j(x, t) = \begin{cases} j_0(x) - j_0(x - a(t)), & t < 0, \\ j_0(x), & t > 0. \end{cases}$$

The transform of this function is

$$(16.26)\qquad j^{\pm}(k, t) = [1 - \exp\{\mp ika(t)\}]j^{\pm}_0(k) \qquad \text{for} \qquad t < 0;$$

hence the function $g^{\pm}(k, t)$ approaches the function

$$(16.27)\qquad g^{\pm}(k, 0) = \pm i\omega \int_{-\infty}^{0} \exp\{\mp it\omega\}[1 - \exp\{\mp ika(t)\}]\, dt\; q^{\pm}_0(k)$$

as t approaches zero through negative values.

The factor of $q^{\pm}_0(k)$ in (16.29) is finite, if e.g. the function $a(t)$ has a continuous derivative $\dot{a}(t)$ for which $\int_{-\infty}^{0} |\, \dot{a}(t)\, |\, dt < \infty$. It could be shown that this factor vanishes with ω strongly enough to make $g^{\pm}(k, 0)$ belong to $\mathfrak{K}$ if $(q, \Omega q) < \infty$ even if $(q, q) = \infty$. Therefore, the process of removing sinks from sources just described could also be used to switch on a source distribution with $(q, q) = \infty$.

Switching Off

So far we considered the effects of switching *on* a source distribution; now we shall consider the effects of *switching a source distribution on and then off again.* We assume that the source distribution behaves in the same way for

$t \to \infty$ as for $t \to -\infty$. We may formulate this assumption by requiring that the functions $g^{\pm}(t, s)$ given by (16.5) possess limits

$$g_\infty^{\pm}(s) = \pm i\Omega^s \int_{-\infty}^{\infty} \exp\{\mp it\Omega^s\} q^{\pm}(s, t)\, dt \qquad \text{as} \qquad t \to \infty \tag{16.28}$$

in the sense that

$$\| g(t) - g_\infty \| \to 0 \qquad \text{as} \qquad t \to \infty. \tag{16.29}$$

Therefore the transition $U(t) = T_0\{g(t)\}$, see (16.9), also possesses a limit

$$U(\infty) = T_0\{g_\infty\}. \tag{16.30}$$

Since $T_0\{g(t)\} \to 1$ as $t \to -\infty$ by (16.10), it is clear that *the asymptotic transition operator* $U(\infty) = T_0\{g_\infty\}$ *is simply Heisenberg's scattering operator for the present problem.* For, the scattering operator is characterized by the property that it transforms the limit of the interaction state $\Phi_w(t)$ as $t \to -\infty$ into the limit of the interaction state as $t \to \infty$. Thus, while no scattering operator exists when the source distribution is not switched off, as shown in Section 15, *there exists a scattering operator when the source distribution is switched on and off.*

We say that *a source distribution is switched off infinitely slowly* beginning at the time $t = t^+$ if

$$g^{\pm}(s, t) - g^{\pm}(s, t^+) = \hat{q}^{\pm}(t^+, s, t^+) - \hat{q}^{\pm}(t, s, t^+), \qquad t^+ < t, \tag{16.31}$$

in analogy to (16.14).

Let us consider the case in which the source distribution is switched on infinitely slowly up to a certain time, kept constant for a while, and then switched off infinitely slowly. Note that a period during which the sources are kept constant may be incorporated in any period of infinitely slow switching. Hence we may set $t^+ = t^-$. Comparing (16.30) with (16.14) we find that (16.14) holds for arbitrary $t > t^-$. In this case, letting $t \to \infty$, we derive relation

$$g_\infty^{\pm}(s) = 0 \tag{16.32}$$

from (16.31). As a consequence, the scattering operator $U(\infty)$ reduces to the identity. In other words, *no scattering takes place asymptotically if the sources are switched on infinitely slowly, kept constant for a while, and then switched off again infinitely slowly.*

Lorentz Invariant Formulation

The problem treated in Sections 15, 16 referred to a particular Lorentz frame with respect to which the source distribution is constant. No Lorentz frame, however, is distinguished if, as in the present section, any time dependent source distribution is considered. Although it is true that the definition of the scattering operator given here involves reference to a particular Lorentz frame, one should expect this operator to be actually independent of the choice of this frame. In order to prove that this is the case, one may either carry out the transformation to a new frame, or one may express the scattering operator in a form which shows its invariance under Lorentz transformation explicitly.

Elegant Lorentz invariant expressions for the scattering operator and for the asymptotic transition probabilities were derived by Glauber [34], using ideas of Schwinger; see also [31]. We proceed to show that Glauber's expressions are identical with ours.

We use the first form (14.23) for the operator $T\{g_\infty\}$ and accordingly write the scattering operator (16.30) in the form

$$U(\infty) = \exp\{(g_\infty, A_0) - (A_0, g_\infty)\}. \tag{16.33}$$

We insert formula (16.28) for $g^{\pm}_\infty(s)$ and express $q^{\pm}(s, t)$ in terms of $j^{\pm}(s, t)$ using (16.2). Using (16.6) we obtain

$$\begin{aligned}(g_\infty, A) - (A, g_\infty) &= i\int (2\Omega^S)^{-1/2}\int [A_0^-(s)\exp\{-it\Omega^S\}j^+(s, t) \\ &\qquad + A_0^+(s)\exp\{it\Omega^S\}j^-(s, t)]\,dt\,dm(s) \\ &= i\int (2\Omega^S)^{-1/2}\int_{-\infty}^{\infty} [j^+(s, t)A^-(s, t) \\ &\qquad + A^+(s, t)j^-(s, t)]\,dt\,ds + i\gamma\end{aligned}$$

with

$$\gamma = i\iint_{-\infty}^{\infty} [g^+(s, t)\nabla_t g^-(s, t) - g^-(s, t)\nabla_t g^+(s, t)]\,dt\,dm(s).$$

Now we use $s = x$ as quantum variable so that $j^{\pm}(x, t) = j(x, t)$ and employ relation (8.29). We find

$$U(\infty) = \exp\{i\gamma\}\exp\left\{i\iint_{-\infty}^{\infty} j(x, t)\Xi(x, t)\,dt\,dx\right\}. \tag{16.34}$$

The second factor is Lorentz invariant. In order to show that the constant γ is also Lorentz invariant we express γ in terms of $j(x, t)$, in the form

$$\begin{aligned}\gamma = i\int (2\Omega^x)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\operatorname{sgn}(t'' - t')\exp\{-it'\Omega^X\}j(x, t') \\ &\cdot \exp\{it''\Omega^X\}j(x, t'')\,dt'\,dt''\,dx\end{aligned}$$

or

$$\begin{aligned}\gamma = -\frac{1}{2}\iint_{-\infty}^{\infty}\iint_{-\infty}^{\infty} &j(x', t')\Gamma(x'' - x', t'' - t') \\ &\cdot j(x'', t'')\,dx''\,dt''\,dx'\,dt',\end{aligned} \tag{16.35}$$

using the improper function

$$\Gamma(x,\ t) = \operatorname{sgn}(t)(\Omega^X)^{-1} \sin t\Omega^X\ \delta(x). \tag{16.36}$$

Note that the contribution from the real part of the exponential function drops out since γ is obviously real.

The function $\Gamma(x,\ t)$ evidently satisfies the differential equation

$$\Delta^2\Gamma + (\Omega^X)^2\Gamma = 0 \qquad \text{for} \qquad t \neq 0.$$

Letting t approach zero through positive or negative values we see that $\Gamma(x,\ t)$ satisfies the initial conditions

$$\Gamma(x,\ 0) = 0, \qquad \nabla_t\Gamma(x,\ 0) = \delta(x) \qquad \text{and} \qquad -\delta(x),$$

respectively. This fact is sufficient to identify the function $\Gamma(x,\ t)$ with the function $\Gamma_+(x,\ t) + \Gamma_-(x,\ t)$ in the notation of Section 3. This function is simply the fundamental solution of Jordan and Pauli denoted by $\overline{\Delta}(x,\ t)$ in Schwinger's work. Its Lorentz invariant character is thus evident. Formula (16.34) together with (16.35), (16.36) agrees with that of Glauber.

Another quantity which is independent of the Lorentz frame is the probability $P(N_\infty\ , n \mid N_{-\infty}\ , 0)$ that n particles will be found eventually, i.e. as $t \to \infty$, if the field was in a vacuum originally i.e. for $t \to -\infty$. From (16.30) and (14.36) we find

$$P(N_\infty\ , n \mid N_{-\infty}\ , 0) = n\text{¡}(g_\infty\ , g_\infty)^n \exp\{-(g_\infty\ , g_\infty)/2\}. \tag{16.37}$$

By (16.28) and (16.2), the quantity $(g_\infty\ , g_\infty)$ is given by

$$(g_\infty\ , g_\infty) = \iint_{-\infty}^{\infty} \int_{-\infty}^{\infty} j(x,\ t')(2\Omega^X)^{-1} \exp\{i(t'' - t')\Omega^X\} j(x,\ t'')\ dt'\ dt''\ dx$$

or, since only the real part of the exponential function contributes, by

$$(g_\infty\ , g_\infty) = \frac{1}{2} \iint_{-\infty}^{\infty} j(x',\ t')\Gamma^{(1)}(x' - x'',\ t' - t'') \cdot j(x'',\ t'')\ dx'\ dt'\ dx''\ dt'' \tag{16.38}$$

with

$$\Gamma^{(1)}(x,\ t) = (\Omega^X)^{-1} \cos t\Omega^X\ \delta(x). \tag{16.39}$$

The improper function $\Gamma^{(1)}(x,\ t)$ evidently satisfies the differential equation

$$\Delta_t^2\Gamma^{(1)} + (\Omega^X)^2\Gamma^{(1)} = 0 \qquad \text{for} \qquad t \neq 0$$

and the initial condition

$$\Gamma^{(1)}(x,\ 0) = (\Omega^X)^{-1}\ \delta(x), \qquad \nabla_t\Gamma^{(1)}(x,\ 0) = 0.$$

From these facts one can deduce that $\Gamma^{(1)}$ is the function

$$\Gamma^{(1)}(x, t) = (2\pi^2)^{-1}\mu \operatorname{sgn}[t^2 - |x|^2]\,|t^2 - |x|^2|^{-1/2}Y_1(\mu\,|t^2 - |x|^2|^{1/2}),$$

also denoted by $\Delta^{(1)}(x, t)$. For $\mu = 0$, in particular, we have

$$\Gamma^{(1)}(x, t) = (2\pi^2)^{-1}[|x|^2 - t^2]^{-1}, \qquad \mu = 0,$$

cf. (4.10). Incidentally, $-\frac{1}{2}\pi\Gamma^{(1)}(x, t)$ is Hadamard's fundamental solution.

Since the function $\Gamma^{(1)}(x, t)$ is evidently Lorentz invariant the same is true of the number (g_∞, g_∞). Formula (16.37) together with (16.38) agrees with that of Glauber.

17. *Modified Vacuum States*

Extending the terminology introduced in Section 14, we call any state Φ_g a "modified vacuum state" if it possesses the N-representation.

$$\Phi_g \leftrightarrow \{[n\text{¡}]^{1/2} g_n^-(s)_n \exp\{-(g, g)/2\}\}. \tag{17.1}$$

Here

$$g_n^-(s)_n = g^-(s_1) \cdots g^-(s_n) \tag{17.2}$$

and $g^-(s)$ is any function in the space $\mathfrak{K}$, i.e.

$$(g, g) = \int |g(s)|^2\,dm(s) < \infty.$$

We shall also say that the state Φ_g is "generated" by the function g. We do not, however, imply any more that this function $g^-(s)$ is related to a source distribution. Nevertheless, we shall speak of the "modified" number operator M and the associated "modified" creation and annihilation operators

$$A^{\pm}(s) - g^{\pm}(s).$$

One may describe a modified vacuum state by saying that in such a state each particle is prepared in the same way, and that this preparation is characterized by the function $g^-(s)$ considered as the probability amplitude of a single particle. The number of these single particles, however, is not determined in such a state. The probability of finding n single particles,

$$P(N, n \mid M, 0) = n\text{¡}(g, g)^n \exp\{-(g, g)\} \tag{17.3}$$

follows Poisson's law.

In previous sections we were concerned with the probabilities that certain numbers of particles would be found and with the expectation values of the number of particles. In the present section we shall derive more detailed information, confining ourselves, however, to the modified vacuum state.

We shall first calculate *the expected number of particles for which the ob-*

servable S has values s in a region $\mathcal{R}$ of the s-space. We denote this number by $W_{\mathcal{R}}$. To evaluate this number we first consider any state Φ, represented as

$$\Phi \leftrightarrow \{\psi_n(s)_n\}.$$

The probability that there are n particles altogether of which the first ρ are in $\mathcal{R}$, while the remaining $n - \rho$ are in the complementary region $\mathcal{R}^*$, is evidently

$$\int_{\mathcal{R}} \int_{\mathcal{R}^*} \mid \psi_n(s)_\rho(s')_{n-\rho} \mid^2 dm(s)_\rho \, dm(s')_{n-\rho} \, .$$

The probability that any ρ particles are in $\mathcal{R}$, and the remaining ones in $\mathcal{R}^*$ is

$$n!\rho\text{¡}(n - \rho)\text{¡} \int_{\mathcal{R}} \int_{\mathcal{R}^*} \mid \psi_n(s)_\rho(s')_{n-\rho} \mid^2 dm(s)_\rho \, dm(s')_{n-\rho} \, .$$

The expected number of particles is, therefore,

$$W_{\mathcal{R}} = \sum_n \sum_\rho n!(\rho - 1)\text{¡}(n - \rho)\text{¡} \int_{\mathcal{R}} \int_{\mathcal{R}^*} \mid \psi_n(s)_\rho(s')_{n-\rho} \mid^2 dm(s)_\rho \, dm(s')_{n-\rho}$$

$$= \sum_n n \sum_\sigma (n - 1)!\sigma\text{¡}(n - 1 - \sigma)\text{¡}$$

$$\cdot \int_{\mathcal{R}} \int_{\mathcal{R}^*} \mid \psi_n(s)_{\sigma+1}(s')_{n-1-\sigma} \mid^2 dm(s)_{\sigma+1} \, dm(s')_{n-1-\sigma} \, ,$$

or

$$W_{\mathcal{R}} = \sum_n n \int_{\mathcal{R}} \int \mid \psi_n(s')_{n-1}(s) \mid^2 dm(s')_{n-1} \, dm(s). \tag{17.4}$$

Note that the integration with respect to s is extended over the region $\mathcal{R}$ while the integration with respect to $(s'_1, \cdots, s'_{n-1})$ is extended over the whole $(s'_1, \cdots, s'_{n-1})$-space. Expression (17.4) may be written symbolically in the form

$$W_{\mathcal{R}} = \int_{\mathcal{R}} dw(s) \tag{17.5}$$

where $dw(s)$ is defined by

$$dw(s) = \sum_n n \int \mid \psi_n(s')_{n-1}(s) \mid^2 dm(s')_{n-1} \, dm(s). \tag{17.6}$$

We say $dw(s)$ is the expected number of particles having values of S in a cell of content $dm(s)$ located at s, or even simpler, of particles "having the value" $S = s$.

The symbolic expressions

$$dw(s) = \langle A^+(s)A^-(s)\rangle_\Phi \, dm(s) = \| A^-(s)\Phi \|^2 \, dm(s) \tag{17.7}$$

could also be verified formally.

We can now easily calculate the number $dw(s)$ of particles expected if the field is in the modified vacuum state Φ_g. Inserting into (17.6) the functions ψ_n given by (17.1) we find

$$dw(s) = \sum_n (n - 1)\text{¡}(g, g)^{n-1} \mid g(s) \mid^2 dm(s) \exp \{-(g, g)\}$$

or

$$dw(s) = \mid g(s) \mid^2 dm(s). \tag{17.8}$$

This noteworthy formula for the number of particles with $S = s$ in the state Φ_g gives a simple interpretation of the modulus $\mid g(s) \mid$ of the function $g^{-}(s)$ which characterizes the modified vacuum state.

Other expectation values may be derived from formula (17.8). Let the observable $\Lambda = \lambda(S)$ be a function of the observable S. Then one may determine the amount of the quantity Λ carried by the particles expected to have the value $S = s$. This amount is evidently $\lambda(s)\, dw(s)$ in general, and

$$\lambda(s)\, dw(s) = \lambda(s) \mid g(s) \mid^2 dm(s) \tag{17.9}$$

when the field is in the modified vacuum state Φ_g.

Let us, for example, choose the momentum $s = k$ as quantum variable and the energy $\Omega = \omega(k)$ as observable. The amount of intrinsic energy Ω of the particles expected to have the momentum k is found to be

$$\omega(k) \mid g(k) \mid^2 dk,$$

if the field is in the modified vacuum state Φ_g.

Influence of a Source Distribution

We now proceed to investigate how these expected values vary when the field is under the influence of a source distribution. In particular, we assume: the field was in a state of vacuum originally, i. e. at $t = -\infty$, the sources having been switched on so as to attain the distribution $j(x)$ at the time $t = 0$, and this distribution is kept constant from then on. The function $g(s, t)$ which "generates" the state of the field at the time t is given by

$$g^{-}(s, t) = g^{-}(s) + q^{-}(s) - \hat{q}^{-}(t, s) \tag{17.10}$$

in accordance with (16.12), (16.13). The function $g^{-}(s)$ "generates" the state at the time $t = 0$; it embodies the effect of the switching on process. Consequently, the expected number of particles having $S = s$ at the time t is

$$dw(s, t) = \mid g(s) + q(s) - q(t, s) \mid^2 dm(s), \tag{17.11}$$

or

$$dw(s, t) = dw_g(s) + dw_{in}(s, t) + dw_{sp}(s, t), \tag{17.12}$$

with

$$dw_g(s) = |\, g(s)\,|^2\, dm(s), \tag{17.13}$$

$$dw_{in}(s,\, t) = 2\, \mathcal{R}e\, g^+(s) \mid [q^-(s) - q^-(t,\, s)] \mid dm(s), \tag{17.14}$$

$$dw_{sp}(s,\, t) = |\, q(t,\, s) - q(s)\,|^2\, dm(s). \tag{17.15}$$

The last term, which is independent of the initial state, may be ascribed to spontaneous processes, the middle term to induced processes.

In case the quantum variable s is the momentum k we have

$$\exp\,\{\pm it\Omega^S\} = \exp\,\{\pm it\omega\}, \qquad \omega^2 = \mu^2 + |\, k\,|^2.$$

Hence, we find

$$dw_{sp}(k,\, t) = 4 \sin^2 (\omega t/2)\, |\, q(k)\,|^2\, dk. \tag{17.16}$$

Therefore, the number of particles with the momentum k spontaneously emitted per unit time is expected to be

$$\nabla_t\, dw_{sp}(k,\, t) = 2 \sin \omega t\, |\, q(k)\,|^2\, dk. \tag{17.17}$$

Thus it is expected that particles of every momentum are emitted and reabsorbed spontaneously; the flux of these particles is expected to oscillate with the circular frequency ω.

Suppose the source distribution consists of two congruent but opposite distributions described by a function $j_0(x)$ and placed at the points $x = \pm a$,

$$j(x) = j_0(x - a) - j_0(x + a). \tag{17.18}$$

Then we have

$$q^{\pm}(k) = 2 \sin ka\; q_0^{\pm}(k) \tag{17.19}$$

in obvious notation. Accordingly, the number of spontaneously emitted or reabsorbed particles depends on the distance of the two source distributions in such a way that it vanishes if $2ak$ is an integral multiple of 2π. It may be of some interest to formulate this condition without assuming that Planck's constant h has the value 2π. In terms of the momentum $p = kh/2\pi$ the condition is that the product $2ap$ be an integral multiple of h.

In order to illustrate the expression for the induced emission we assume, for example, that the field at the time $t = 0$ was so prepared that nearly all particles have the momentum k_0 or $-k_0$. Specifically, we assume that the function $g(k)$ characterizing the modified vacuum state at that time is given by

$$g^{\pm}(k) = b\beta(k - k_0) + b\beta(k + k_0). \tag{17.20}$$

Here the real valued function $\beta(k \mp k_0)$ should be an approximation to Dirac's δ-function $\delta(k \mp k_0)$. As before, we assume the source distribution for $t > 0$ to be given by $j(x) = j_0(x - a) - j_0(x + a)$.

From the second relation (17.14) we find

$$dw_{in}(k, t) = -4b[\beta(k - k_0) + \beta(k + k_0)] \cdot \sin ka\, \mathscr{I}m\, \bar{q}_0(k)[1 - \exp\{it\omega\}]. \tag{17.21}$$

Only particles with a momentum nearly equal to $\pm k_0$ are therefore expected to be emitted or re-absorbed. No induced process is expected to take place if the number $2k_0a$ is nearly an integer multiple of 2π, i.e. if $2p_0a$ is nearly an integral multiple of h.

Of course, one could also calculate *the number of particles expected to be found at a point x in space.*[6] We confine ourselves to the calculation of the contribution due to spontaneous emission assuming that the source distribution is concentrated at the origin

$$j(x) = e\delta(x). \tag{17.22}$$

This distribution is, of course, improper but it can be considered as approximating a proper distribution. According to (17.18), the spontaneously emitted number of particles expected to be found at the point x is

$$dw_{sp}(x, t) = e^2 \mid [1 - \exp\{it \mid \nabla_x \mid\}] \mid 2\nabla_x \mid^{-3/2} \delta(x) \mid^2 dx. \tag{17.23}$$

We have

$$[1 - \exp\{it \mid \nabla_x \mid\}] \mid \nabla_x \mid^{-3/2} \delta(x)$$

$$= (2\pi)^{-3} \int [1 - \exp\{it \mid k \mid\}] \mid k \mid^{-3/2} \exp\{ikx\}\, dk$$

$$= (2\pi)^{-2}\, 2 \mid x \mid^{-1} \int_0^\infty [1 - \exp\{it \mid k \mid\}] \mid k \mid^{-1/2} \sin(\mid k \mid \mid x \mid) d \mid k \mid$$

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[6] Throughout this exposition we have designated the observable X associated with a single particle as the "position" of the particle. The reason was that the observable X is connected with the momentum K by the Fourier transformation and obeys the commutation law $XK - KX = i$. Nevertheless the observable X does not possess all the properties which are commonly implied when one speaks of the position of a particle.

First of all, the observable X depends on the Lorentz frame to which it is referred; cf. Pauli [1, p. 260], see also [30]. Suppose, for example, the field consists of one particle in a state in which measurement of X at the time $t = 0$ would yield approximately the value zero. If, instead, the "position" X' at the time $t' = 0$ is measured with reference to a different Lorentz frame which passes through $x = 0$, $t = 0$, an arbitrarily large value of $\mid X' \mid$ may be found with a positive probability.

Secondly, the procedures commonly employed for the measurement of the position of a particle of matter would not yield the value of the observable X for a boson because a boson field does not interact with external forces in the same way as matter.

The notion of "number of particles to be found at a point in space" should, therefore, be understood with these qualifications in mind.

$$= \pi^{-2} \mid x \mid^{-1} \int_0^\infty [1 - \exp \{it\sigma^2\}] \sin (\mid x \mid \sigma^2) \, d\sigma$$

$$= \pi^{-2}(2i)^{-1} \mid x \mid^{-1} 2^{-1/2} \int_0^\infty [(1 + i) \exp \{-\mid x \mid \tau^2\}$$

$$- (1 - i) \exp \{-\mid x \mid \tau^2\} - (1 + i) \exp \{-[t + \mid x \mid]\tau^2\}$$

$$+ (1 + i) \exp \{-[t - \mid x \mid]\tau^2\}] \, d\tau$$

if $t > \mid x \mid$. Hence

$$\begin{aligned} &[1 - \exp \{it \mid \nabla_x \mid\}] \mid \nabla_x \mid^{-3/2} \delta(x) \\ &= (2\pi)^{-3/2} \mid 2x \mid^{-1} [2 \mid x \mid^{-1/2} - (1 - i)\{[t + \mid x \mid]^{-1/2} - [t - \mid x \mid]^{-1/2}\}. \end{aligned} \tag{17.24}$$

For large values of $t \mid x \mid^{-1}$ we therefore have

$$\begin{aligned} &[1 - \exp \{it \mid \nabla_x \mid\}] \mid \nabla_x \mid^{-3/2} \delta(x) \\ &\qquad \sim (2\pi)^{-3/2}[\mid x \mid^{-3/2} + \tfrac{1}{2}(1 - i)t^{-3/2}], \end{aligned} \tag{17.25}$$

whence

$$dw_{sp}(x, t) \sim \mid 4\pi x \mid^{-3} [1 + \mid x \mid^{3/2} t^{-3/2}]e^2 \, dx \qquad \text{for} \qquad t \gg \mid x \mid. \tag{17.26}$$

A refined expression for this number of particles expected to be found at the point x can be derived from formula (17.25).

Probability Distribution of the Energy

In the present section we have mainly been concerned with the number of particles in a modified vacuum state for which the observable S is expected to have the value s. We proceed to discuss a different question which can be answered in connection with modified vacuum states.

Let

$$B = [\lambda(S)] = \int A^+(s)\lambda(s)A^-(s) \, dm(s)$$

be an operator which results through biquantization from the single particle operator $\Lambda = \lambda(S)$. We then ask for the probability $P(\beta)$ that a measurement of the observable B will yield a value less than a number β when the field is in the modified vacuum state Φ_g. For example, we may ask for *the probability $P(\epsilon)$ that the energy $H = [\omega(K)]$ of the field will be found to have a value below a number ϵ.*

We suppose the state of the field is represented as

$$\Phi \leftrightarrow \{\psi_n(s)_n\}.$$

We further denote by $\mathcal{R}_\beta^n$ the region in the $(s_1, \cdots, s_n)$-space in which

$$\lambda(s_1) + \cdots + \lambda(s_n) < \beta.$$

Then the desired probability is evidently,

$$P(\beta) = \sum_n \int_{\mathcal{R}_{n,\beta}} |\psi_n(s)_n|^2\, dm(s)_n\,. \tag{17.28}$$

If Φ is the modified vacuum state Φ_g so that

$$\psi_n(s)_n = (n!)^{1/2} g^-(s_1) \cdots g^-(s_n) \exp\{-(g, g)/2\},$$

we find

$$P(\beta) = \sum_n n! \int_{\mathcal{R}_{n,\beta}} |g(s_1)|^2 \cdots |g(s_n)|^2\, dm(s)_n \exp\{-(g, g)\}. \tag{17.29}$$

In a manner well known from the theory of probability, this expression can be cast in a different form which, under certain circumstances, permits us to evaluate it more easily.

We use the discontinuity factor

$$(2\pi i)^{-1} \int_{\mathcal{L}} \exp\{z[\beta - \lambda(s_1) - \cdots - \lambda(s_n)]\} z^{-1}\, dz = D(s)_n \tag{17.30}$$

which has the value

$$D(s)_n \begin{array}{lll} = 1 & \text{if} & (s)_n \text{ is in } \mathcal{R}_\beta^n\,, \\ = 0 & \text{if} & (s)_n \text{ is outside of the closure } \overline{\mathcal{R}_\beta^n}\,. \end{array}$$

Here the path $\mathcal{L}$ extends in the right half plane from $z = x_0 - i\infty$ to $z = x_0 + i\infty$, $x_0 > 0$.

We insert this factor in formula (17.28) and interchange the order of integration and summation. Moreover we make use of the identity

$$\int \exp\{-z\lambda(s_1) - \cdots - z\lambda(s_n)\} |g(s_1)|^2 \cdots |g(s_n)|^2\, dm(s)_n = Z^n(z\lambda),$$

involving the n-th power of the "partition function"

$$Z(z\lambda) = \int \exp\{-z\lambda(s)\} |g(s)|^2\, dm(s). \tag{17.31}$$

Then we find

$$P(\beta) = (2\pi i)^{-1} \int_{\mathcal{L}} \sum_n n! Z^n(z\lambda) \exp\{\beta z\} z^{-1}\, dz$$

or

$$P(\beta) = (2\pi i)^{-1} \int_{\mathcal{L}} \exp\{Z(z\lambda) + \beta z\} z^{-1}\, dz. \tag{17.32}$$

We shall derive this formula in a different manner in Section 20 on the basis of the occupation number representation. There we shall also discuss possibilities of approximate evaluation of its right member. We have derived this formula here because we wanted to show that this can be done easily on the basis of the particle representation.

BIBLIOGRAPHY TO PART III

[27] Born, M., *Das Adiabatenprinzip in der Quantenmechanik*, Zeitschrift für Physik, Volume 40, 1926, pp. 167–192.

Born, M., and Fock, V., *Beweis des Adiabatensatzes*, Zeitschrift für Physik, Volume 51, 1928, pp. 165–180.

[28] Bloch, F., and Nordsieck, A., *Note on the radiation field of the electron*, Physical Review, Volume 52, 1937, pp. 54–58.

Nordsieck, A., *The low frequency radiation of a scattered electron*, Physical Review, Volume 52, 1937, pp. 59–62.

[29] Heisenberg, W., *Die "beobachtbaren Groessen" in der Theorie der Elementarteilchen*, Zeitschrift für Physik, Volume 120, 1943, pp. 513–538; *Die "beobachtbaren Groessen" in der Theorie der Elementarteilchen, II*, *ibid.*, pp. 673–702.

[30] Le Couteur, K. J., *The theory of neutral particles*, Proceedings of the Cambridge Philosophical Society, Volume 44, Part 2, April, 1948.

[31] Thirring, W., and Touschek, B., *A covariant formulation of the Bloch-Nordsieck method*, Philosophical Magazine, Ser. 7, Volume 42, 1951, p. 244.

[32] Snyder, H. S., *Remarks concerning the adiabatic theorem and the S-matrix*, Physical Review, Volume 83, 1951, pp. 1154–1159.

[33] Feynman, R. P., *An operator calculus having applications in quantum electrodynamics*, Physical Review, Volume 84, 1951, pp. 108–128.

[34] Glauber, R. J., *Some Notes on Multiple-Boson Processes*, Physical Review, Volume 84, 1951, pp. 395–400.

SUPPLEMENTARY REFERENCES TO PARTS I AND II

[35] Valatin, J. G., *Sur la Seconde Quantification*, Le Journal de Physique et le Radium, Volume 12, February 1951, pp. 131–141.

Valatin, J. G., *Sur la Seconde Quantification. II. Théorie du Positron*, Le Journal de Physique et le Radium, Volume 12, April 1951, pp. 542–549.

[36] Cook, J. M., *The Mathematics of Second Quantization*, Proceedings of the National Academy of Sciences, Volume 37, No. 7, July 1951, pp. 417–420.

Part IV. Occupation Number Representation and Fields of Different Kinds

18. *Occupation Number Representation*

An occupation number representation of the states of a field is customarily employed only for fields enclosed in a box. To every eigenstate Ψ_r of the energy of single particles one assigns the number n_r of particles which are in this state. The label r, which runs over the integer values $r = 1, 2, \cdots$, assumes the role of the "quantum variable" here generally denoted by s. As explained in Section 7, the states Φ of the field may be represented by a probability amplitude $\psi'(n_1, n_2, \cdots)$ considered as a function of the occupation numbers $n_1, n_2, \cdots$.

In the course of our exposition we have asserted several times that a similar occupation number representation is always possible, no matter which quantum variable s is chosen. In particular, s could be any continuous quantum variable, such as the momentum or the position of a single particle. In such a case the occupation number, which we shall denote by $\nu(s)$, depends on a continuous variable but may assume only the integral values $\nu = 0, 1, 2, \cdots$.

We had also remarked several times that the formulation of general occupation number representations is more cumbersome than that of particle representations. Nevertheless, valuable information can be derived from such general occupation number representations. For this reason we shall describe such representations in the present Part IV.

One might be inclined to derive the desired representations involving a continuous variable s by limiting processes from a representation involving a discrete quantum variable; for example, one might consider a field which extends over the whole space as the limit of fields enclosed in a box. We shall not proceed in this manner. Instead we shall employ partitions of the s-space in cells and consider states in which the probability of finding each cell occupied by a certain number of particles is independent of the distribution of the particles within the cells. We shall consider refinements of such partitions and then perform a limit process.

One might also be inclined to introduce a kind of occupation number density as a function of s and to determine the total number of particles as an integral of such a density over the whole s-space. Such a procedure, however, would result in very clumsy formulas. Instead, we shall employ the occupation number $\nu(s)$ itself, i.e. the number of particles at the place s; the total number of particles is given as a sum $\sum_s \nu(s)$ to be extended over all points s of the s-space. Of course, this sum is finite only if a finite number of places are occupied.

It should be noted that, in general, there are no proper states of the field in which the value of ν at each point s can be predicted with certainty; nor is, in general, a probability assigned to each distribution of occupation numbers $\nu(s)$. Such probabilities will rather be assigned to manifolds of occupation functions $\nu(s)$ which comprise functions $\nu(s)$ obtained from each other by varying slightly the places s that are occupied.

Instead of saying that a state Φ of the field is represented by a function of the infinitely many variables $\nu(s)$, we prefer to say that the representer of Φ is a "functional" $f(\nu)$ of the "occupation function" $\nu(s)$. The inner product of two states Φ and $\Phi^{(1)}$ will be expressed in the form[1]

$$\sum_\nu \bar{f}(\nu) f^{(1)}(\nu) \prod_s \nu(s)¡\,[dm(s)]^{\nu(s)},$$

in which the summation $\sum_\nu$ refers to all possible occupation functions $\nu(s)$ and the infinite product $\prod_s$ to all possible points s. To be sure, this expression is only a symbol for the result of a certain limit process; we find it, however, advantageous to employ this and similar symbolic expressions. The evaluation of expectation values and probabilities can be greatly simplified by formal procedures using such symbols. Of course, these formal operations can be justified rigorously.

Complete proofs of the statements made in the following sections will in general be omitted or merely indicated when they would involve lengthy arguments of a standard pattern; only in a few cases will proofs be carried through.

The occupation number representation referred to will be developed in the present section in connection with the particle representation. In Section 19 we shall develop a different type of occupation number representation. In analyzing this different type of representation we shall realize that there are cases in which it is not equivalent with a particle representation. In other words, we shall be able to describe fields whose states do not admit a particle representation and, accordingly, do not possess a vacuum state. It will be shown that in a certain sense such fields without vacuum occur in nature.

In spite of the non-existence of a vacuum various expectation values and probabilities can be calculated for such fields. This will be shown in Section 20.

In Section 21 we shall show how our considerations, so far concerning only boson fields, can be carried over to fermion fields. In this connection we shall refer to Dirac's idea of an electron field which may consist of a finite number of electrons with a positive energy while every state with a negative energy is occupied. With the aid of the two kinds of fields—the one with and the other without vacuum—we shall be able to give Dirac's idea, in a somewhat modified form, a precise mathematical meaning. This fact might be of interest although the notion of electrons with negative energy has really never been used explicitly by physicists.

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[1] As in Part III, we use the abbreviation $n¡ = 1/n!$ for $n = 0, 1, \cdots$ and $n¡ = 0$ for $n = -1, -2, \cdots$.

Particle and Occupation Representation

In the particle representation the state is represented by a system of functions $\psi_n(s)_n$, which may be considered as one function of the infinitely many variables $n, s_1, s_2, \cdots$. The possible values of these variables are the possible values of the observables $N, S_1, S_2, \cdots$. The observable N is the total number of particles composing the field; the observable S_r is the quantity S associated with the r-th particle. The symmetry of the function $\psi_n(s)_n$ insures that these particles are indistinguishable.

We recall that the symbol S stands for any complete set of observables associated with a single particle. This observable may have a continuous, discrete, or mixed spectrum. The nature of the spectrum of S is embodied in the measure function $m(\mathfrak{S})$ associated with S. On occasion we shall refer to the point eigenvalues of S; they are values s^0 of s such that the set which consists of exactly this point has a positive measure, $m(s^0)$.

Instead of the observables $N, S_1, S_2, \cdots$, the occupation numbers may be employed as observables. The occupation number $N(s)$ associated with a point s is the number of particles for which the quantity S has the value s. The possible values of $N(s)$ will be denoted by $\nu(s)$. At each point s the number ν may assume the values $\nu = 0, 1, 2, \cdots$; we therefore call it "integer-valued". The totality of the occupation numbers associated with all points s is given by the observable N as a function $N(s)$ of s, which will be termed the "occupation distribution". The possible values of this manifold of observables $N(s)$ are given by functions $\nu(s)$, which will be called "occupation functions" or simply "occupations". Specifically, we shall use the term "occupation" for any integer-valued function $\nu(s)$ which equals zero except for a finite number of values s. The total number of particles in an occupation is then finite:

$$\sum_s \nu(s) < \infty. \tag{18.1}$$

This requirement does not at all mean that we restrict ourselves to considering states of the field in which the expectation value of the total number of particles is finite. What is implied is that we shall assign a probability amplitude only to occupations which involve a finite total number of particles.

The notion of probability amplitude is not quite adequate for the present purpose since in general the field does not possess a discrete manifold of proper eigenstates for the set of operators $N(s)$. Instead we consider functions—or rather functionals—$f(\nu)$ of the occupation $\nu(s)$ as representers of the states of the field. We must set up a Hilbert space of these "occupation functionals" with respect to an appropriate unit form assigned to them. Furthermore we must represent the observables of the field by appropriate functional operators. The probability that any such observable has a value in a given range when the field is in a given state can then be expressed with the aid of the representer $f(\nu)$ of the state.

The functionals $f(\nu)$ to be introduced will first be expressed in terms of the representers $\psi_n(s)_n$ of the particle representation.

Let $\nu(s)$ be any occupation. From the function $\nu(s)$ we shall derive a number n and values $s_1, s_2, \cdots, s_n$; the value of $f(\nu)$ for this occupation $\nu(s)$ will then be defined in terms of the value of $\psi_n(s)_n$ for these values of $n, s_1, \cdots, s_n$. We first set

$$n = \sum_s \nu(s). \tag{18.2}$$

This number is finite by (18.1). Consequently, there are at most n distinct values $s = s^{(1)}, s^{(2)}, \cdots$ for which $\nu(s) \neq 0$. Using the notation

$$\nu_1 = \nu(s^{(1)}), \qquad \nu_2 = \nu(s^{(2)}), \qquad \cdots$$

we set

$$s_1 = \cdots = s_{\nu_1} = s^{(1)}, \qquad s_{\nu_1+1} = \cdots = s_{\nu_1+\nu_2} = s^{(2)}, \qquad \cdots .$$

Clearly n values $s_1, \cdots, s_n$ are thus determined since $\nu_1 + \nu_2 + \cdots = n$. With the aid of the given function $\psi_n(s)_n$ we now assign to the occupation $\nu(s)$ the number

$$f(\nu) = [n!]^{1/2}\psi_n(s)_n . \tag{18.3}$$

In this way, the functional $f(\nu)$ is defined for all occupations satisfying condition (18.2) provided the particle representers $\psi_n(s)_n$ are given. The factor $[n!]^{1/2}$ is merely chosen for convenience.

Vice versa, we can express the functions $\psi_n(s)_n$ in terms of the functional $f(\nu)$. Let any values of $n, s_1, \cdots, s_n$ be given. For every value of s let $\nu(s)$ be the number of values among $s_1, \cdots, s_n$ which equal s. Thus an occupation is described. Then the value of $\psi_n(s)_n$ for the given values of $n, s_1, s_2, \cdots$ is evidently obtained by inverting (18.3),

$$\psi_n(s)_n = [n!]^{1/2} f(\nu). \tag*{(18.03)$'$}$$

As an example we mention the functional $f(\nu) = f_{\text{vac}}(\nu)$ which represents the vacuum state Φ_{vac}; it is, evidently,

$$f_{\text{vac}}(\nu)\begin{cases} = 1 & \text{if} \quad \nu(s) \equiv 0, \\ = 0 & \text{if} \quad \nu(s) \not\equiv 0. \end{cases} \tag{18.4}$$

Similarly, the representers $f(\nu)$ of states in which there are n particles are characterized by the condition that they vanish unless $\sum_s \nu(s) = n$.

The description, given above, of the transformation of the two representations into each other cannot be taken literally. For, the value of a function $\psi_n(s)_n$ at a particular set of values of the variables $s_1, \cdots, s_n$ may be altered without altering the state represented; in fact, two functions $\psi_n(s)_n$ may serve as representers for the same state if they differ in a set of measure zero. Such a change in the function $\psi_n(s)_n$ evidently induces a change in the functional $f(\nu)$

Occupation Functionals

Before giving the transformation (18.3) a precise meaning we must describe a manifold of occupation functionals $f(\nu)$ independently of the functions $\psi_n(s)_n$ and define for them a unit form (f, f). To this end we shall first introduce functionals which depend only on the total number of particles in certain cells of a partition of the s-space. From these special functionals we shall obtain all functionals $f(\nu)$ by a limit process. This process is nothing but a generalization of one of the possible processes by which the integral of a finite number of variables can be defined.

A "partition" $\mathcal{P}$ of the s-space is given by a subdivision of this space into distinct rectangular—bounded or semi-infinite—cells. These cells need not be sets of the same dimension as the s-space; in fact, a single point s may serve as a cell. The region formed by all bounded cells in $\mathcal{P}$, the "finite part" of $\mathcal{P}$, will be denoted by $\mathcal{R} = \mathcal{R}^{\mathcal{P}}$, the cells composing it by $\mathcal{C}_k$, $k = 1, 2, \cdots$. The region formed by the remaining cells will be denoted by $\mathcal{C}_* = \mathcal{C}_*^{\mathcal{P}}$.

With reference to such a partition we form for every occupation $\nu(s)$ the numbers

$$\begin{aligned} \nu_k &= \sum_{s \text{ in } \mathcal{C}_k} \nu(s), \qquad k = 1, 2, \cdots, \\ \nu_* &= \sum_{s \text{ in } \mathcal{C}_*} \nu(s). \end{aligned} \tag{18.5}$$

We then say that a functional $f(\nu)$ is "compatible" with the partition $\mathcal{P}$ if it depends only on the values ν_k, $k = 1, 2, \cdots$, and ν_*, and if it vanishes unless $\nu_* = 0$. Accordingly, there is a function $f^{\mathcal{P}}(\nu_1, \nu_2, \cdots, \nu_*)$ such that

$$f(\nu) = f^{\mathcal{P}}(\nu_1, \nu_2, \cdots, \nu_*) \tag{18.6}$$

and

$$f^{\mathcal{P}}(\nu_1, \nu_2, \cdots, \nu_*) = 0 \quad \text{unless} \quad \nu_* = 0. \tag{18.6$'$}$$

We now define the inner product of two functions f, $\hat{f}$ which are compatible with the partition $\mathcal{P}$. To this end we introduce the measure

$$m_k = \int_{\mathcal{C}_k} dm(s), \qquad k = 1, 2, \cdots \tag{18.7}$$

of the cells $\mathcal{C}_k$. We then set

$$(f, \hat{f}) = \sum_{\nu_1, \nu_2, \cdots} \bar{f}^{\mathcal{P}}(\nu_1, \nu_2, \cdots, 0)\hat{f}^{\mathcal{P}}(\nu_1, \nu_2, \cdots, 0) \prod_k \nu_k! m_k^{\nu_k}. \tag{18.8}$$

The summation indicated by the symbol $\sum_{\nu_1,\nu_2,\cdots}$ extends over $\nu_1 = 0, 1, 2, \cdots$, $\nu_2 = 0, 1, 2, \cdots, \nu_3 = 0, 1, 2, \cdots$; the subscript k in the product symbol $\prod_k$ extends over all values of k to which cells $\mathcal{C}_k$ are assigned. We further stipulate that

$$m_k^{\nu_k} = 1 \qquad \text{if} \qquad \nu_k = 0$$

even if $m_k = 0$. The series (18.8) converges if the series

$$\| f \|^2 = (f, f) \tag{18.8$'$}$$

and $(\hat{f}, \hat{f})$ converge, which we assume.

A partition $\mathcal{P}'$ is called a "refinement" of the partition $\mathcal{P}$ if each cell of $\mathcal{P}'$ is contained in a cell of $\mathcal{P}$ and the region $\mathcal{R}'$ associated with $\mathcal{P}'$ contains the region $\mathcal{R}$ associated with $\mathcal{P}$ so that $\mathcal{C}_*'$ is contained in $\mathcal{C}_*$. Clearly, a function $f(\nu)$ which is compatible with $\mathcal{P}$ is also compatible with $\mathcal{P}'$. We maintain that the value of the inner product $(f, \hat{f})$ of two functions compatible with $\mathcal{P}$ is unchanged if it is taken with respect to the refined partition $\mathcal{P}'$.

To see that this is so, let $\mathcal{C}_{rk}'$ and $\mathcal{C}_{r*}'$ denote all cells of $\mathcal{P}'$ which compose $\mathcal{C}_k$ and $\mathcal{C}_*$ respectively, so that

$$\sum_r \nu'_{rk} = \nu_k , \qquad \sum_r m'_{rk} = m_{k*} ,$$

and

$$\sum_r \nu'_{r*} + \nu'_* = \nu_* .$$

We observe that the inner product of f and $\hat{f}$ with reference to $\mathcal{P}'$ can be written in the form

$$\sum_{\nu_1,\nu_2,\cdots,\nu_*} \bar{f}^{\mathcal{P}}(\nu_1, \nu_2, \cdots, \nu_*)\hat{f}^{\mathcal{P}}(\nu_1, \nu_2, \cdots, \nu_*)$$
$$\cdot \prod_k \sum_{\nu'_{1k},\nu'_{2k},\cdots} \prod_r \nu'_{rk}\text{¡}[m'_{rk}]^{\nu'_{rk}} \sum_{\nu'_{1*},\nu'_{2*},\cdots} \prod_r \nu'_{r*}\text{¡}[m'_{r*}]^{\nu'_{r*}} .$$

Now we first note that

$$\sum_{\nu'_{1k},\nu'_{2k},\cdots} \prod_r \nu'_{rk}\text{¡}[m'_{rk}]^{\nu'_{rk}} = \nu_k\text{¡}[m_k]^{\nu_k},$$

by the binomial theorem. Secondly, we note that the contribution from ν'_{1*}, $\nu'_{2*}, \cdots$ to the inner product vanishes unless $\nu'_{1*} = \nu'_{2*} = \cdots = 0$; and in that case the contribution evidently consists exactly of the factor 1. The statement that the expression (18.8) does not change if a partition is refined thus follows.

Since evidently any two partitions have a common refinement, any two functionals compatible with two partitions are compatible with a common partition. Therefore, the manifold of functions which are compatible with some partition forms a linear space in which the inner product is defined by (18.8).

One of the axioms to be satisfied by the elements of an inner product space is that vanishing of the unit form, $(f, f) = 0$, implies vanishing of the element, $f = 0$. This condition is not necessarily satisfied by the compatible functionals. For example, if there is a cell $\mathfrak{C}_r$ with measure zero, the unit form vanishes for any functional $f^{\mathcal{P}}(\nu_1, \cdots, \nu_*)$ which vanishes unless $\nu_k = 0$, $k \neq r$, $\nu_* = 0$. In order to avoid this occurrence one might assign a new entity to each class of equivalent functionals, terming two functionals $f_1(\nu)$, $f_2(\nu)$ "equivalent" if $(f_1 - f_2, f_1 - f_2) = 0$. Instead, we find it more convenient to say that two such functionals are "equal", f_1 "$=$" f_2, or simply $f_1 = f_2$. With this stipulation the manifold of compatible functionals remains a linear space. This identification of two functionals is analogous to the identification of two quadratically integrable functions of a finite number of variables if they differ only in a set of measure zero.

This linear space of compatible functionals will be extended to a complete linear space, a Hilbert space of functionals; these functionals will then be established as representers of states.

To complete this linear space one need only assign an "ideal element" to every Cauchy sequence of compatible functionals, i.e. to every sequence of functionals $f_\sigma(\nu)$ for which $(f_\sigma - f_\tau, f_\sigma - f_\tau) \to 0$ as $\sigma, \tau \to \infty$. These ideal elements will also be called "occupation functionals" and denoted by $f(\nu)$. The manifold of all—original and ideal—elements established by such a process of completion forms a Hilbert space[2]. The Hilbert space of all—compatible and ideal—occupation functionals $f(\nu)$ will be denoted by $\mathfrak{B}$.

Naturally, it would be of interest to investigate the possibility of identifying the ideal occupation functionals with functionals in the proper sense, i.e. with assignments of complex numbers f to occupations $\nu(s)$. For our purposes, however it is not necessary to do so; for, it is possible to calculate the desired probabilities and expectation values using only compatible and ideal functionals.

For the inner product of two occupation functionals $f(\nu)$ and $\hat{f}(\nu)$ in $\mathfrak{B}$, now defined, we adopt the symbolic notation

$$(f, \hat{f}) = \sum_\nu \bar{f}(\nu)\hat{f}(\nu) \prod_s \nu(s)!\,[dm(s)]^{\nu(s)} \tag{18.10}$$

in order to indicate that the inner product originated from the expression (18.8) valid for functionals compatible with partition $\mathcal{P}$. We should look at expression (18.10) in the following way: For every occupation $\nu(s)$ we form the infinite product $\prod_s \nu(s)!\,[dm(s)]^{\nu(s)}$ in which the variable s extends over the whole s-space. We multiply the result, which depends on the choice of $\nu(s)$, by the value which the functional $\bar{f}(\nu)\hat{f}(\nu)$ has for the occupation $\nu(s)$. Finally we take the sum of the values of the product $\bar{f}(\nu)\hat{f}(\nu) \prod_s \nu(s)!\,[dm(s)]^{\nu(s)}$ which result from all possible choices of the occupation $\nu(s)$. Of course, this description is not to be taken literally, but it is a helpful guide in performing formal operations which can be justified rigorously, as will be indicated later on.

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[2] The simple process of completing an "inner product space" to a Hilbert space—analogous to the process of completing the set of rational numbers by introducing irrational numbers—is mentioned only casually in the introductions to Hilbert space [41, p. 17]. In the following we shall make frequent use of this, and related, completion processes.

Correspondence between Occupation Functionals and Particle Representers

We proceed to establish a one-to-one correspondence between the "functionals" $f(\nu)$ in $\mathfrak{B}$ and the particle representers of the states Φ, thus establishing a rigorous basis for the formal arguments given above.

Let us first assume that the functional $f(\nu)$ is compatible with a partition $\mathcal{P}$. Then the assignment (18.3)′ can be carried out. The resulting function $\psi_n(s)_n$ is evidently constant whenever any of the variables $s_1, \cdots, s_n$ varies over a cell of the partition. The value of this function when s_1 lies in a cell $\mathfrak{C}_{k_1}$, s_2 in a cell $\mathfrak{C}_{k_2}, \cdots, s_n$ in $\mathfrak{C}_{k_n}$, will be denoted by $\psi^{\mathcal{P}}_{k_1,\cdots,k_n}$. Note that $\psi_n(s)_n = 0$ if one of the values $s_1, s_2, \cdots$, or s_n lies in $\mathfrak{C}_*$ since $f(\nu) = 0$ unless $\nu_* = 0$. The unit-form of the state Φ represented by this function can therefore be expressed as

$$(18.11)\qquad \begin{aligned}(\Phi, \Phi) &= \sum_n \int |\, \psi_n(s)_n \,|^2\, d(s)_n \\ &= \sum_n \sum_{k_1,\cdots,k_n} |\, \psi^{\mathcal{P}}_{k_1,\cdots,k_n} \,|^2\, m_{k_1} \cdots m_{k_n} .\end{aligned}$$

To any set of numbers, $\nu_1, \nu_2, \cdots$ with $\nu_1 + \nu_2 + \cdots = n$ there are $n!\nu_1¡\nu_2¡ \cdots$ sets of numbers $k_1, k_2, \cdots, k_n$ of which ν_1 are equal to 1, ν_2 equal to 2, and so on. Consequently, there are just $n!\nu_1¡\nu_2¡ \cdots$ terms $|\, \psi^{\mathcal{P}}_{k_1,k_2,\cdots,k_n} \,|^2$ which are equal to $n¡\, |\, f(\nu_1, \nu_2, \cdots, 0) \,|^2$ by virtue of (18.3)′. The expression (18.11) for (Φ, Φ) can therefore be written as

$$(\Phi, \Phi) = \sum_{\nu_1,\nu_2,\cdots} \nu_1¡\nu_2¡\, |\, f^{\mathcal{P}}(\nu_1, \nu_2, \cdots, 0) \,|^2\, [m_1]^{\nu_1}[m_2]^{\nu_2} \cdots ;$$

hence, by (18.9), as

$$(18.12)\qquad (\Phi, \Phi) = (f, f).$$

Now, let $f(\nu)$ be any functional in $\mathfrak{B}$ and $f_\alpha(\nu)$ be a sequence of functionals compatible with partitions $\mathcal{P}_\alpha$ such that

$$\|\, f_\alpha - f \,\| \to 0, \qquad \text{as} \qquad \alpha \to \infty;$$

then, also

$$\|\, f_\alpha - f_\beta \,\| \to 0 \qquad \text{as} \qquad \alpha, \beta \to \infty .$$

By virtue of (18.10) the relation

$$\|\, \Phi_\alpha - \Phi_\beta \,\| \to 0 \qquad \text{as} \qquad \alpha, \beta \to \infty$$

holds for the states Φ_α, Φ_β which correspond to f_α, f_β. Accordingly, the states Φ_α converge to a limit state Φ which is represented by the "functional" $f(\nu)$; on the other hand, Φ is also represented by functions $\psi_n(s)_n$. Thus a definite set of functions $\psi_n(s)_n$ is assigned to the functional $f(\nu)$. This assignment will be indicated symbolically by formula (18.3)′.

In a similar way a functional $f(\nu)$ in $\mathfrak{B}$ can be assigned to every function $\psi_n(s)_n$ of $n, s_1, s_2, \cdots$ which represents a state Φ. We need only approximate $\psi_n(s)_n$ by functions $\psi_n^{(\alpha)}(s)_n$ which are constant when any of the variables $s_1, s_2, \cdots$ runs over a cell $\mathcal{C}$ of a partition $\mathcal{P}^\alpha$ and vanish whenever any of these variables lies in $\mathcal{C}_*^\alpha$. We then employ formula (18.3) to assign to $\psi_n^{(\alpha)}(s)_n$ a functional $f_\alpha(\nu)$ which is compatible with $\mathcal{P}^\alpha$. The identity $\| f_\alpha - f_\beta \| = \| \Phi_\alpha - \Phi_\beta \|$ implies the existence of a functional $f(\nu)$ in $\mathfrak{B}$ such that $\| f_\alpha - f_\beta \| \to 0$. This functional is then assigned to the function $\psi_n(s)_n$ and thus represents the state Φ.

Representation by Occupation Functionals

The representation of the state Φ by the functional $f(\nu)$ in $\mathfrak{B}$ will be indicated by

$$\Phi \leftrightarrow f(\nu). \tag{18.13}$$

If a definite value can be ascribed to the functional $f(\nu)$ when $\nu(s)$ is a particular occupation $\nu_0(s)$, this value $f(\nu_0)$ may be interpreted as the amplitude of the probability that a measurement of the occupation distribution in the state Φ yields the result $\nu(s) = \nu_0(s)$. This interpretation, however, requires some qualification. If one could speak properly of the probability that a measurement of the observable $N(s)$ yields the result $\nu_0(s)$, the expression

$$f(\nu_0)\Big[\prod_s \nu_0(s)!\,[dm(s)]^{\nu_0(s)}\Big]^{1/2},$$

rather than $f(\nu_0)$, should be regarded as the amplitude of this probability, in keeping with (18.10); the value of the functional $f(\nu_0)$ itself might, therefore, be called the "relative" probability amplitude.

Properly one can speak of the probability that $N(s)$ has the "value" $\nu_0(s)$ only if the occupation distribution $N(s)$ has a proper eigenstate with the "eigenvalue" $\nu(s) = \nu_0(s)$. Then the representer of this eigenstate, a functional $f_0(\nu)$ which vanishes when $\nu(s) \neq \nu_0(s)$, belongs to the Hilbert space $\mathfrak{B}$. In general, however, there is no such functional in $\mathfrak{B}$ and in that case no state exists in which the occupation has the definite value $\nu_0(s)$. Nevertheless, one can in general speak of the probability Pr $\{y(N) \text{ in } \mathcal{I}\}$ that a measurement of an observable $y(N)$ which is a functional of $N(s)$ yields a result which lies in an interval I of the y-axis. More generally, one can speak of the expectation value $\langle y(N)\rangle_\Phi$ of such an observable $y(N)$ when the field is in a state Φ. We shall define and evaluate such expectation values later on.

Different Forms of the Occupation Representation

The occupation number representation of states which we have introduced depends essentially on the choice of the quantum variable s. For example, if the field is enclosed in a box, we may choose s to be a label characterizing the eigenstates of the energy of a single particle also assumed to be enclosed in the box. The special form to which the occupation functionals reduce in this case will be discussed below.

In general, however, the occupation number representation which we have introduced is not restricted to cases in which the quantum variable s is discrete. This variable may, for example, stand for the position of a single particle or for its momentum if the field extends over the infinite space. In this case the notion of an integer valued function, $\nu(s)$, of a continuous variable s must be used. It may, perhaps, be preferable to use the function

$$\int_{-\infty}^{s_1} \cdots \int_{-\infty}^{s_l} \nu(s')\, dm(s_1') \cdots dm(s_l')$$

where l is the dimension of the s-space and $s = \{s_1, \cdots, s_l\}$. It has also been suggested to use instead of $\nu(s)$ the symbolic function $\nu(s)/dm(s)$ as "occupation density"; the representation of the creation operator, see $(18.17)^+$, would then assume a simpler form, but other formulas would become very clumsy. In any case, we regard it as most natural to use the occupation $\nu(s)$ itself as the independent variable of our functionals.

The role of the occupation function ν of a continuous variable s is also illuminated by the following fact, which will be proved in Section 21. Call "effective" every occupation $\nu(s)$ which assumes the value 0 or 1 (but not a value > 1) at every place s which is not a point eigenvalue of S. Furthermore, call "effective" every functional in $\mathfrak{B}$ which vanishes for every non-effective occupation. Then every functional in $\mathfrak{B}$ "equals" an effective one. Consequently, if s is not a point eigenvalue of S, the terms involving values $\nu(s) > 1$ do not contribute to the expression (18.10). In other words, *the probability of an occupation distribution is not affected by multiple occupations of points s, which are not point eigenvalues.* One may also say that *the oscillator assigned to such a value of s is effectively excited at most to the first level.* Similar statements have been made in the literature, see e.g. [14]. However, there seems hardly to be an occasion to make use of this simplification.

Occupation Functions of a Discrete Variable

We shall now discuss the simple form to which the occupation number representation reduces when S has a pure point spectrum. We denote the eigenvalues of S by s^r, $r = 1, 2, \cdots$; the measure $m(\mathfrak{C})$ of a cell $\mathfrak{C}$ in the s-space then consists of the sum of the contributions $m_r = m(s^r)$ of all eigenvalues s^r in the cell,

$$m(\mathfrak{C}) = \sum_{s^r \text{ in } \mathfrak{C}} m_r .$$

We confine ourselves to the case in which each finite cell contains only a finite number of eigenvalues. Incidentally, if s is a label of the eigenstates of single particles in case the field is enclosed in a box, one may simply identify s with r.

It is natural to expect that only the values

$$n_r = \nu(s^r)$$

of the occupation function at the eigenvalues s^r and the functionals

$$f(\nu) = f^0(n_1, n_2, \cdots)$$

of the values of n_r, $r = 1, 2, \cdots$ need be considered. These functionals must, of course, vanish if infinitely many among the numbers n_1, n_2, $\cdots$ differ from zero. Each term in the infinite series

$$(18.10)^0 \qquad (f^0, f^0) = \sum_{n_1, n_2, \cdots} | f^0(n_1, n_2, \cdots) |^2 \prod_{r=1,2,\cdots} n_r! m_r^{n_r}$$

is then defined. Here the summation sign $\sum$ and the product sign $\prod$ may be understood in the literal sense. The functionals f^0 are further restricted by the condition $(f^0, f^0) < \infty$. Since the manifold of all finite sets of numbers $n_1, n_2, \cdots$ is countable, it is clear that the manifold of such functionals f^0 forms a Hilbert space $\mathfrak{B}^0$.

It can now be stated that it is possible to restrict oneself to the space $\mathfrak{B}^0$; for, in a sense to be explained, this space is equivalent with the Hilbert space $\mathfrak{B}$ of functionals $f(\nu)$ introduced before.

In order to indicate how this can be shown we call an occupation $\nu(s)$ "effective" if it vanishes except for a finite number of eigenvalues s^r; we call a functional $f(\nu)$ "effective" if it vanishes for every non-effective occupation $\nu(s)$. (The term "effective" here is different from, though analogous to, the same term used above.) Every effective functional $f(\nu)$ evidently corresponds to a functional $f^0(n_1, n_2, \cdots)$ as introduced above. One could then verify that every functional $f(\nu)$ in $\mathfrak{B}$ is "equal" to an effective functional when equality of two functionals f_1, f_2 means $\| f_1 - f_2 \| = 0$ as stipulated above. For effective functionals $f(\nu)$ the unit form (f, f) as defined by (18.10) evidently agrees with the unit form as defined by $(18.10)^0$. Therefore, the Hilbert spaces $\mathfrak{B}$ and $\mathfrak{B}^0$ are identical.

The occupation numbers n_r considered here are those referred to in Section 7. The representation in terms of functionals $f^0(n_1, n_2, \cdots)$—there denoted by $\psi'(n_1, n_2, \cdots)$—is essentially that employed in the literature (see e.g. [19, III, 6]).

It was mentioned earlier that, in general, proper eigenstates for the occupation distribution do not exist. If, however, the quantum variable is discrete, such eigenstates do exist. In fact, every occupation $n_r = n_r^0$ with $\sum_r n_r^0 < \infty$ occurs as an "eigenvalue". The representer $f^0(n_1, n_2, \cdots)$ of the corresponding eigenstate is given by

$$f^0(n_1, n_2, \cdots)\begin{cases} = c & \text{if} \quad n_1 = n_1^0, n_2 = n_2^0, \cdots \\ = 0 & \text{otherwise} \end{cases}$$

where c is any constant $\neq 0$. The unit form

$$(f^0, f^0) = c^2 \prod_r n_r^0!\, m_r^{n_r^0}$$

is finite and positive because of $\sum_r n_r^0 < \infty$; hence this functional f^0 belongs to the space $\mathfrak{B}^0$.

Annihilation and Creation Operators

We proceed to consider the annihilation and creation operators $A^{\mp}(s)$, associated with the particle representation $\Phi \overset{\leftrightarrow}{_N} \{\psi_n(s)_n\}$, and investigate how they are represented when Φ is represented by $f(\nu)$. Let $\gamma^{\mp}(s)$ be any quadratically integrable function. Then the states

$$\int A^{\mp}(s')\gamma^{\mp}(s')\, dm(s')\, \Phi$$

are also represented by functionals in $\mathfrak{B}$ if the state Φ, assumed to admit these operators, is represented by the functional $f(\nu)$. We set

$$(18.14)^{\mp} \qquad \int A^{\mp}(s')\gamma^{\mp}(s')\, dm(s')\, \Phi \underset{\nu}{\leftrightarrow} \int A^{\mp}_{(\nu)}(s')\gamma^{\mp}(s')\, dm(s')\, f(\nu)$$

thus defining the symbolic operators $A^{\mp}_{(\nu)}(s)$ acting on functionals $f(\nu)$.

Let us first assume that the functional $f(\nu)$ is compatible with a partition $\mathcal{P}$. We then express the right member of $(18.14)^-$ in terms of $f(\nu)$. Let $\nu(s)$ be an integer-valued function with $\sum_s \nu(s) = n$, and choose $(s)_n$ such that $\nu(s)$ is the number of values among $s_1, \cdots, s_n$ which equal s. Then, clearly, by (18.3)′,

$$\int A^-_{(\nu)}(s')\gamma^+(s')\, dm(s')\, f(\nu) = [n!]^{1/2}[n+1]^{1/2} \int \psi_{n+1}(s)_n(s')\gamma^+(s')\, dm(s').$$

In order to express the value of $\psi_{n+1}(s)_n(s')$ in terms of $f(\nu)$ we follow the arguments that led to (18.3). The number of values among $s_1, \cdots, s_n, s'$ which equal s is evidently

$$\nu(s) + \delta_{s'}(s).$$

Here we use the Kronecker function

$$(18.15) \qquad \delta_{s'}(s) = \begin{cases} 1 & \text{if} \quad s = s' \\ 0 & \text{if} \quad s \neq s'. \end{cases}$$

Consequently, by (18.3)′,

$$\psi_{n+1}(s)_n(s') = [(n+1)!]^{1/2} f(\nu + \delta_{s'})$$

and hence

$$(18.16)^- \qquad \int A^-_{(\nu)}(s')\gamma^+(s')\, dm(s')\, f(\nu) = \int f(\nu + \delta_{s'})\gamma^+(s')\, dm(s').$$

We express this relation symbolically as

$$(18.17)^- \qquad A^-_{(\nu)}(s')f(\nu) = f(\nu + \delta_{s'}).$$

Next we express the right member of $(18.14)^+$ in terms of $f(\nu)$. From (18.3) we deduce

$$\int A^+_{(\nu)}(s')\gamma^-(s')\, dm(s')\, f(\nu) = [n!]^{1/2} n^{1/2} \,\mathrm{Sy}\, \psi_{n-1}(s)_{n-1}\gamma^-(s_n)$$

where again $(s)_n$ is so chosen that $\nu(s)$ is the number of values in $(s)_n$ which equal s. The number of values in $(s)_{n-1}$ which equal s is then $\nu(s) - \delta_{s_n}(s)$. By $(18.3)'$, therefore,

$$\psi_{n-1}(s)_{n-1} = [(n-1)!]^{1/2} f(\nu - \delta_{s_n}).$$

Consequently,

$$\mathrm{Sy}\, \psi_{n-1}(s)_{n-1}\gamma^-(s_n) = [(n-1)!]^{1/2} n^{-1} \sum_{\kappa=1}^{n} f(\nu - \delta_{s_\kappa})\gamma^-(s_\kappa).$$

Since the number of values among the $s_1, \cdots, s_n$ which equal s' is just equal to $\nu(s')$, this expression becomes

$$\mathrm{Sy}\, \psi_{n-1}(s)_{n-1}\gamma^-(s_n) = [(n-1)!]^{1/2} n^{-1} \sum_{s'} \nu(s')f(\nu - \delta_{s'})\gamma^-(s'),$$

whence

$$(18.16)^+ \qquad \int A^+_{(\nu)}(s)\gamma^-(s)\, dm(s)\, f(\nu) = \sum_s \nu(s)f(\nu - \delta_s)\gamma^-(s).$$

Using the symbolic relationship

$$(18.18) \qquad \delta_{s'}(s) = \delta(s - s')\, dm(s')$$

between Kronecker's and Dirac's delta functions, we may express relation $(18.16)^+$ symbolically as

$$(18.17)^+ \qquad A^+_{(\nu)}(s')f(\nu) = [dm(s')]^{-1}\nu(s')f(\nu - \delta_{s'}).$$

Formulas $(18.16)^\pm$ have been established in the case of a functional $f(\nu)$ which is compatible with a partition $\mathcal{P}$. Such compatible functionals represent states of a certain class. The annihilation and creation operators $\int A^\pm(s)\gamma^\mp(s)\, dm(s)$ are applicable to the states of a larger manifold, $\mathfrak{D}'$, dense in the Hilbert space $\mathfrak{D}$ of all states. Therefore, the operators $\int A^\pm_{(\nu)}(s)\gamma^\mp(s)\, dm(s)$ are applicable to the functionals $f(\nu)$ of the manifold $\mathfrak{B}'$ formed by the representers $f(\nu)$ of states in $\mathfrak{D}'$. It is expedient to use the formulas $(18.16)^\pm$ and $(18.17)^\pm$ to describe these operators symbolically.

The operators $\int A^\mp_{(\nu)}(s)\gamma^\pm(s)\, dm(s)$ have so far been defined in connection with the particle representation. However they should also be defined directly, independently of the particle representation. To this end these operators should first be defined for compatible functionals by (18.16) and then extended by the process of closure; cf. [40]. We do not intend to give the details of this straightforward process.

The symbolic formulas

$$(18.17)^- \qquad A^-_{(\nu)}(s')f(\nu) = f(\nu + \delta_{s'}),$$

$$(18.17)^+ \qquad A^+_{(\nu)}(s')f(\nu) = [dm(s)]^{-1}\nu(s')f(\nu - \delta_{s'})$$

are the natural generalizations of those given by Jordan and Klein (see e.g. [1], [4]) for the case of a discrete quantum variable s. These formulas have a certain elegant appearance, but they do not show clearly on which functionals $f(\nu)$ the operators $A^{\pm}_{(\nu)}$ are applicable. One of the main advantages of the particle representation is that this difficulty does not arise.

Formal Operations

We shall now perform the evaluations of a few inner products and expectation values in a purely formal manner as if the variable s in the expression $\prod_s \nu(s)\,\text{¡}[dm(s)]^{\nu(s)}$ were running over a finite number of values. The validity of this procedure could always be established in the following manner: First one assumes that every functional which occurs is compatible with a partition and that every function of s which occurs is piecewise constant; by this we mean that the function is constant on the bounded cells of a partition and vanishes on the unbounded cells. For such functionals and functions the desired operations are permitted. The validity of these operations for functionals and functions of a more general nature can then be established by limit processes. We shall not carry out details.

Of particular interest will be the functional

$$(18.19) \qquad f_g(\nu) = \prod_s [g^-(s)]^{\nu(s)}$$

generated by a function $g^-(s)$ for which

$$(18.20) \qquad \int |\, g(s)\,|^2\, dm(s) < \infty .$$

For, this functional is the representer of the equidistribution state to be discussed later on.

Here and in the following we use our earlier convention to denote the complex conjugate of a function $g^-(s)$ by $g^+(s)$ and to set

$$|\, g^-(s)\,| = |\, g^+(s)\,| = |\, g(s)\,|.$$

If the function $g^-(s)$ is piecewise constant with reference to a partition $\mathcal{P}$, and hence vanishes in $\mathcal{C}_*$, the functional $f_g(\nu)$ is defined by (18.19) as it stands, provided relation

$$(18.21) \qquad [g^-(s)\,^0 = 1$$

is adopted even if $g^{-}(s) = 0$. Clearly, $f_g(\nu)$ is then compatible with $\mathcal{P}$. In general, $f_g(\nu)$ should be defined as the limit of a sequence of such compatible functionals.

To evaluate the unit form $(f_g\,, f_g)$ we take the following steps:

$$\begin{aligned} |\, f_g(\nu)\, |^2 &= \prod_s |\, g(s)\, |^{2\nu(s)}, \\ (f_g\,, f_g) &= \sum_\nu \prod_s |\, g(s)\, |^{2\nu(s)} \prod_s \nu(s)\text{¡}[dm(s)]^{\nu(s)} \\ &= \sum_\nu \prod_s \nu(s)\text{¡}[|\, g(s)\, |^2\, dm(s)]^{\nu(s)} \\ &= \prod_s \sum_{\nu(s)=0,1,\cdots} \nu(s)\text{¡}[|\, g(s)\, |^2\, dm(s)]^{\nu(s)} \\ &= \prod_s \exp\,\{|\, g(s)\, |^2\, dm(s)\} \\ &= \exp\,\{\sum_s |\, g(s)\, |^2\, dm(s)\} \end{aligned}$$

or

$$(18.22) \qquad (f_g\,, f_g) = \exp\left\{\int |\, g(s)\, |^2\, dm(s)\right\}.$$

Note that in these formal operations no distinction is made between the integral sign $\int$ and the summation sign $\sum$.

It would not be difficult to justify this result by the procedure indicated above.

Another evaluation which will play a role is:

$$(18.23) \qquad \begin{aligned} &\sum_\nu \Big(\sum_s \lambda(s)\nu(s)\Big)\ |\, f_g(\nu)\, |^2 \prod_s \nu(s)\text{¡}[dm(s)]^{\nu(s)} \\ &\qquad = \int \lambda(s)\ |\, g(s)\, |^2\, dm(s)\ \exp\left\{\int |\, g(s)\, |^2\, dm(s)\right\} \end{aligned}$$

It is valid if

$$(18.24) \qquad \int |\, \lambda(s)\, |\ |\, g(s)\, |^2\, dm(s) < \infty.$$

To derive identity (18.23) formally we first assume $\lambda(s) = \delta_{s'}(s)$ and proceed as follows:

$$\begin{aligned} &\sum_\nu \nu(s')\ |\, f_g(\nu)\, |^2 \prod_s \nu(s)\text{¡}[dm(s)]^{\nu(s)} \\ &\qquad = \sum_\nu{}' \prod_{s\neq s'} \nu(s)\text{¡}[|\, g(s)\, |^2\, dm(s)]^{\nu(s)}[\nu(s') - 1]\text{¡}[|\, g(s')\, |^2\, dm(s')]^{\nu(s')}. \end{aligned}$$

The prime in the summation sign $\sum_\nu'$ indicates that the summation is to extend over all integer valued functions $\nu(s)$ for which $\nu(s') \neq 0$. This condition can also be expressed by saying that the function $\nu(s) - \delta_{s'}(s)$ extends over all integer valued functions without restriction. Changing the notation by substituting $\nu(s) + \delta_{s'}(s)$ for $\nu(s)$ we may write the last identity in the form

$$\sum_{\nu} \nu(s') \mid f_0(\nu) \mid^2 \prod_{s} \nu(s)![dm(s)]^{\nu(s)}$$

$$= \sum_{\nu} \prod_{s \neq s'} \nu(s)![| g(s) |^2 dm(s)]^{\nu(s)} \nu(s)![| g(s') |^2 dm(s')]^{\nu(s)+1}$$

$$= \sum_{\nu} \prod_{s} \nu(s)![| g(s) |^2 dm(s)]^{\nu(s)} \mid g(s') \mid^2 dm(s')$$

$$= \mid g(s') \mid^2 dm(s') \exp \int \mid g(s) \mid^2 dm(s).$$

Multiplication by $\lambda(s')$ and summation with respect to s' then yields identity (18.23).

Biquantized Operators

With the aid of formal operations we can derive a simple expression for the operator which acts on functionals $f(\nu)$ and represents the operator

$$[\lambda(S)] = A^{+}\lambda(S)A^{-} = \int A^{+}(s)\lambda(s)A^{-}(s)\, dm(s);$$

we recall that this operator results through biquantization from the single particle operator Λ assumed to be a function $\Lambda = \lambda(S)$ of S, see (10.7). Suppose the state Φ is represented by the functional $f(\nu)$; then, as seen from (18.17)* the state $A^{+}\lambda(S)A^{-}\Phi$ is represented by the functional

$$\int [dm(s')]^{-1}\nu(s')\lambda(s')f(\nu)\, dm(s) = \sum_{s} \lambda(s')\nu(s')f(\nu).$$

In other words, we obtain the representer $A^{+}_{(\nu)}\lambda(S)A^{-}_{(\nu)}$ of the operator $A^{+}\lambda(S)A^{-}$ by simply multiplying by $\sum_{s} \lambda(s)\nu(s)$:

$$A^{+}_{(\nu)}\lambda(S)A^{-}_{(\nu)} = \sum_{s} \lambda(s)\nu(s). \tag{18.25}$$

The number operator $N = A^{+}A^{-}$, in particular, is represented by

$$N_{(\nu)} = \sum_{s} \nu(s); \tag{18.26}$$

the meaning of $\sum \nu(s)$ will be significant throughout this section.

Formula (18.25) for the ν-representers of $A^{+}\lambda(S)A^{-}$ motivates the symbolic notations

$$A^{+}(s)A^{-}(s)\, dm(s) = N(s) \tag{18.27}$$

and

$$A^{+}\lambda(S)A^{-} = \int A^{+}(s)\lambda(s)A^{-}(s)\, dm(s) = \sum_{s} \lambda(s)N(s) \tag{18.28}$$

for the represented operators themselves.

Relation (18.25) may also be interpreted as expressing the fact that the representation of the states Φ by functionals $f(\nu)$ yields the spectral representation of the operators $\sum_s \lambda(s)N(s)$ since application of these operators is represented by multiplication by the functional $\sum_s \lambda(s)\nu(s)$ of $\nu(s)$. One may also say that the representation of Φ by $f(\nu)$ yields the simultaneous spectral representation of the operators $N(s)$, although these operators are improper unless s is a point eigenvalue.

Modified Vacuum State

The functional $f_g(\nu) = \prod_s [g^-(s)]^{\nu(s)}$ considered above, see (18.19), is the occupation number representer of a state Φ_g whose particle representer is

$$\begin{aligned} \psi_n(s)_n &= [n!]^{1/2} g^-(s_1) \cdots g^-(s_n), \qquad && n > 0, \\ &= 1, && n = 0, \end{aligned} \tag{18.29}$$

as is seen from the argument used to derive (18.3)′. Except for a factor $\exp \{\frac{1}{2} \int | g(s) |^2 \, dm(s)\}$, the state Φ_g is that investigated in Part III, see (14.35). It was called a modified vacuum state because it is the vacuum state of the modified creation and annihilation operators

$$B^\pm(s) = A^\pm(s) - g^\pm(s). \tag{18.30}$$

We note that evaluation (18.22), which leads to formula

$$(\Phi_g , \Phi_g) = \exp\left\{\int | g(s) |^2 \, dm(s)\right\}, \tag{18.31}$$

is confirmed by the results of Section 14.

Application of the annihilation operator $A^-_{(\nu)}(s')$, described by (18.17), on the functional $f_g(\nu)$ gives

$$A^-_{(\nu)}(s')f_g(\nu) = g^-(s')f_g(\nu), \tag*{(18.32)$^-$}$$

whence

$$B^-_{(\nu)}(s')f_g(\nu) = 0, \tag{18.33}$$

thus confirming that $f_g(\nu)$ represents the modified vacuum state.

From the description (18.17)$^+$ of the creation operator $A^+_{(\nu)}(s)$ we derive relation

$$g^-(s') \, dm(s')A^+_{(\nu)}(s')f_g(\nu) = \nu(s')f_g(\nu) \tag*{(18.32)$^+$}$$

involving the creation operator $A^+_{(\nu)}$.

In order to make proper relations out of the symbolic relations (18.32) we may introduce functions $\zeta(s)$ for which

$$\int | \zeta(s) |^2 | g(s) |^2 \, dm(s) < \infty, \tag{18.34}$$

and deduce from (18.32) the relations

$$(18.35)^- \qquad \int g^+(s)\zeta(s)A^-_{(\nu)}(s)\, dm(s)\, f_0(\nu) = \int \zeta(s) \mid g(s) \mid^2 dm(s)\, f_0(\nu),$$

$$(18.35)^+ \qquad \int A^+_{(\nu)}(s)\zeta(s)g^-(s)\, dm(s)\, f_0(\nu) = \sum_s \zeta(s)\nu(s)f_0(\nu).$$

In these relations application of the annihilation operator consists in multiplication by a constant given as an integral, while application of the creation operator consists in multiplication by a sum which is linear in $\nu(s)$.

The unit form of the functional $\sum_s \zeta(s)\nu(s)f_0(\nu)$ could easily be evaluated by using formal operations as indicated above. More general evaluations of this kind will be given in Section 20.

Expectation Values

We have seen above, see (18.25) and (18.28), that the biquantized operator

$$A^+\lambda(S)A^- = \sum_s \lambda(s)N(s)$$

is represented by multiplication by $\sum_s \lambda(s)\nu(s)$. The expectation value of the observable $\sum_s \lambda(s)N(s)$ in the modified vacuum state is therefore given by

$$\langle \sum_s \lambda(s)N(s)\rangle_{\Phi_0} = \sum_\nu \sum_s \lambda(s)\nu(s) \mid f_0(\nu) \mid^2 \prod_s \nu(s)!\,[dm(s)]^{\nu(s)}/(f_0\, , f_0).$$

From formulas (18.23), (18.22) we then have

$$(18.37) \qquad \langle \sum_s \lambda(s)N(s)\rangle_{\Phi_0} = \int \lambda(s) \mid g(s) \mid^2 dm(s).$$

In particular, the expected value of the total number of particles is

$$(18.38) \qquad \langle N\rangle_{\Phi_0} = \int \mid g(s) \mid^2 dm(s)$$

in agreement with (14.38).

Symbolically, we can say that the expected number of those particles for which the observable S has the value s is

$$(18.39) \qquad \langle N(s)\rangle_{\Phi_0} = \mid g(s) \mid^2 dm(s),$$

and hence the expected value of the observable $\lambda(S)$ carried by these particles is

$$(18.40) \qquad \langle \lambda(s)N(s)\rangle_{\Phi_0} = \lambda(s) \mid g(s) \mid^2 dm(s).$$

These formulas are in agreement with (17.8) and (17.9). As a matter of fact, the derivation of these formulas on the basis of the occupation number representation appears to be simpler and more natural than that based on particle representation.

Equidistribution State

It is desirable to have a characterization of the state Φ_g which does not refer to the fact that it is a modified vacuum state. We shall term it a "relative equidistribution" state, because, as formula (18.39) indicates, the number of particles is expected to be equally distributed over the s-space relative to the weight $| g(s) |^2 \, dm(s)$. Relative equidistribution states will play a primary role in the subsequent considerations.

19. *Myriotic and Amyriotic Fields*

In the present section we shall first discuss an—apparently minor—generalization of occupation number representations discussed in Section 18. We introduce a measure function $w(\mathbb{S})$, called the "weight", which may differ from the measure function $m(\mathbb{S})$, and then the functional

$$\tilde{f}(\nu) = f(\nu) \exp \{\tfrac{1}{2}W\} \tag{19.1}$$

involving the "total weight"

$$W = \int dw(s); \tag{19.2}$$

here the integration is extended over the whole s-space. Using the symbolic expression (18.10) we may write the unit form of the functional $\tilde{f}(\nu)$ as

$$(\tilde{f}, \tilde{f})^w = \sum_\nu | \tilde{f}(\nu) |^2 \prod_s \nu(s)¡ [dm(s)]^{\nu(s)} \exp \{-W\}. \tag{19.3}$$

Writing the exponential function symbolically as a product,

$$\exp \{W\} = \prod_s \exp \{dw(s)\}, \tag{19.4}$$

we may write relation (19.3) even in the form

$$(\tilde{f}, \tilde{f})^w = \sum_\nu | \tilde{f}(\nu) |^2 \prod_s \nu(s)¡ [dm(s)]^{\nu(s)} \exp \{-dw(s)\}. \tag{19.3$'$}$$

So far it has been tacitly assumed that the total weight W is finite. It is remarkable that this restriction can be dropped. We shall introduce functionals $\tilde{f}(\nu)$ independently of the functionals $f(\nu)$ and without assuming $W < \infty$. We shall define a unit form for these functionals and adopt the symbolic expression (19.3)′ to describe it. In case $W = \infty$, it is not possible to write this expression in the form (19.3) and the functionals $\tilde{f}(\nu)$ are no longer connected by relation (19.1) with functionals $f(\nu)$ of the type considered in Section 18. Nevertheless, it is possible to carry over to the functionals $\tilde{f}(\nu)$ many of the operations introduced in Section 18 for the functionals $f(\nu)$.

In particular, we shall introduce creation and annihilation operators $A^{\mp}_{(\nu)}(s)$ acting on the functionals $\tilde{f}(\nu)$; they will be described symbolically by

$$A^{-}_{(\nu)}(s')\tilde{f}(\nu) = \tilde{f}(\nu + \delta_{s'}) \tag{19.4$^-$}$$

$$A^{+}_{(\nu)}(s')\tilde{f}(\nu) = [dm(s')]^{-1}\nu(s')\tilde{f}(\nu - \delta_{s'}), \tag{19.4$^+$}$$

i.e. in the same way as the operators $A^{\mp}_{(\nu)}(s)$ acting on the functionals $f(\nu)$ introduced in Section 18, see $(18.17)^{\mp}$. Also the operators

$$\int A^{+}_{(\nu)}(s)\lambda(s)A^{-}_{(\nu)}(s)\, dm(s) = \sum_{s} \lambda(s)\nu(s), \tag{19.5}$$

cf. (18.25), will be introduced for certain functions $\lambda(s)$. In case $W = \infty$, however, it is not possible to introduce the operator

$$N_{(\nu)} = \int A^{+}_{(\nu)}(s)A^{-}_{(\nu)}(s)\, dm(s) = \sum_{s} \nu(s), \tag{19.6}$$

although it is possible to introduce the operator

$$N_{(\nu)}(\mathfrak{C}) = \int_{\mathfrak{C}} A^{+}_{(\nu)}(s)A^{-}_{(\nu)}(s)\, dm(s) = \sum_{s \text{ in } \mathfrak{C}} \nu(s), \tag{19.7}$$

associated with any cell $\mathfrak{C}$ provided that the weight $w(\mathfrak{C}) = \int_{\mathfrak{C}} dw(s)$ of the cell is finite.

The significance of the possibility of introducing functionals $\tilde{f}(\nu)$ associated with different weights will become apparent when we consider these functionals as representers of states Φ of a field,

$$\Phi \underset{\nu}{\leftrightarrow} \tilde{f}(\nu) \tag{19.8}$$

assuming that the unit form (Φ, Φ) is given by the expression $(19.3)'$

$$(\Phi, \Phi) = (\tilde{f}, \tilde{f})^{w}. \tag{19.9}$$

The operators $A^{\mp}_{(\nu)}(s)$ acting on functionals $\tilde{f}(\nu)$ are then the representers of creation and annihilation operators $A^{\mp}(s)$ acting on states Φ of the field. For, evidently, these operators $A^{\mp}(s)$ obey the commutation law

$$\begin{aligned} &[A^{-}(s'), A^{+}(s'')] = \delta(s' - s''), \\ &[A^{-}(s'), A^{-}(s'')] = [A^{+}(s'), A^{+}(s'')] = 0. \end{aligned} \tag{19.10}$$

The operator

$$N(\mathfrak{C}) = \int_{\mathfrak{C}} A^{+}(s)A^{-}(s)\, dm(s), \tag{19.11}$$

represented by the operator $N_{(\nu)}(\mathfrak{C})$, furthermore corresponds to the number of particles in the cell $\mathfrak{C}$, and the operator

$$N = \int A^{+}(s)A^{-}(s)\, dm(s), \tag{19.12}$$

represented by $N_{(\nu)}$, corresponds to the total number of particles in the field.

Suppose a field for which operators $A^{\pm}(s)$ are defined possesses a representation (19.8) such that the operators $A^{\pm}(s)$ are represented by $A^{\pm}_{(\nu)}(s)$. Then the number operator N does not exist in case the total weight W involved is infinite.

To make this statement about the non-existence of the number operator more specific, we may consider the expectation value

$$\langle N(\mathfrak{C})\rangle_{\Phi} = \sum_{\nu} | \tilde{f}(\nu) |^{2} \Big(\sum_{s \text{ in } \mathfrak{C}} \nu(s)\Big) \prod_{s} \nu(s)!\,[dm(s)]^{\nu(s)} \exp\{-dw(s)\} \tag{19.13}$$

of the number $N(\mathfrak{C})$ of particles in a cell $\mathfrak{C}$ when the field is in a state Φ represented by the functional $\tilde{f}(\nu)$. The statement then is that in case $W = \infty$ this expected value increases indefinitely when the cell $\mathfrak{C}$ tends to cover the whole s-space, whatever the state Φ may be. Symbolically we express this statement by the formula

$$\langle N\rangle_{\Phi} = \infty \qquad \text{if} \qquad W = \infty, \tag{19.14}$$

although properly speaking the operator N is not defined for $W = \infty$.

Fields with an infinite total weight W which admit a representation as described will be called "myriotic"; they will be called "amyriotic" if they possess such a representation with $W < \infty$. The fact expressed by formula (19.14) may also be expressed by saying that the total number of particles is expected to be infinite in every state of a myriotic field. For this reason the term "myriotic" was chosen.

Fields which admit particle representations and hence occupation number representations of the type described in Section 18 will be called "ordinary". It will turn out that amyriotic fields are ordinary and vice versa. Therefore, the terms ordinary and amyriotic may eventually be interchanged.

The impossibility of defining a number operator N for myriotic fields implies that *myriotic fields do not possess particle representations*, and hence, in particular, that *myriotic fields do not possess vacuum states*; this is perhaps the most striking property of such fields. On the other hand, as we shall show later on, *myriotic fields possess equidistribution states*, similar to the equidistribution states of ordinary fields described in the last subsection of Section 18.

The existence of myriotic and amyriotic fields throws light on the question—raised at the starting point of this exposition—whether or not there exist operators $\Xi(x)$, $\Pi(x)$ which obey the commutation laws or, what is equivalent, whether or not creation and annihilation operators $A^{\pm}(x)$ really exist. The answer was affirmative: a realization of operators $A^{\pm}(x)$ could be given with the aid of the particle representation of the field. It was stated at the end of Section 5 that this realization is not the only one and that there are other realizations not derivable from a particle representation. The results of the present section confirm this statement, since myriotic fields do not admit particle representations.

The representation (19.8) of the states of the field by functionals $\tilde{f}(\nu)$ of the occupation $\nu(s)$ evidently depends on the choice of the quantum variable s. The property of being myriotic or amyriotic, however, does not depend on this choice. Fields of both kinds can be associated with any quantum variable s, since a weight function $w(S)$ with $\int dw(s) = \infty$ and $\int dw(s) < \infty$ can always be found. Of course, we assume that the space of single particle states has an infinite dimension so that the spectrum of S does not consist of a finite number of values.

We may, for example, let the variable s be a label for the energy eigenstates of particles enclosed in a box so that $\nu(s)$ is given by a sequence $n_1, n_2, \cdots$ of integers. Functionals $\tilde{f}(\nu)$ associated with a finite or infinite total weight may then be introduced just as well as if s were a continuous variable. In other words, even *a field enclosed in a box may be myriotic or amyriotic.* Myriotic fields in a box will be discussed at the end of this section.

Whether a field is myriotic or amyriotic depends on the operators $A^*(s)$ connected with a representation (19.8). Properly, we should therefore speak of an "A-representation." It may happen, that other creation and annihilation operators $B^*(s)$ are defined for the field and that the field admits a B-representation of the type (19.8). It is then not excluded that the field is myriotic with respect to the operators B, but ordinary with respect to the operators A.

The question naturally arises whether or not the fields occurring in nature are myriotic or amyriotic; but this question is perhaps rather academic. States of both kinds of fields are constructed by limit processes; although these processes differ they both start with states described in terms of the distribution of a finite number of particles over a finite number of cells. The different limit processes may be appropriate for different experimental situations. The question, however, whether or not a field which is ordinary with respect to operators $A^*(s)$ is also ordinary with respect to other operators $B^*(s)$ appears to have a physical significance. A situation in which the field is ordinary with respect to operators $A^*(s)$ and myriotic with respect to operators $B^*(s)$ arises in connection with the infrared catastrophe treated in Part III. The operators $A^*(s)$ may be referred to the unmodified field and the operators $B^*(s)$ to the modified field or vice versa. This case will be discussed in more detail later on. It is interesting to note that the vacuum state with respect to the operators $A^*(s)$ appears as a relative equidistribution state with respect to the operators $B^*(s)$. In any case, this example shows that in a certain sense *myriotic fields do occur in nature.*

Functionals of the Second Type

Before defining the functionals $\tilde{f}(\nu)$ referred to above we shall introduce another class of functionals $\phi(\nu)$. Eventually we shall express the functionals $\tilde{f}(\nu)$ in terms of the functionals $\phi(\nu)$ in the form

$$\tilde{f}(\nu) = \phi(\nu) \prod_{s} [v^{-}(s)]^{\nu(s)}. \tag{19.15}$$

Here $v^-(s)$ is a function which is connected with the weight differential $dw(s)$ by the relation

$$|\, v(s)\,|^2\, dm(s) = dw(s); \tag{19.16}$$

other requirements on $v(s)$ will be specified later on. At first we shall be solely concerned with the functionals $\phi(\nu)$.

The nature of these functionals $\phi(\nu)$ will depend only on the nature of the weight function $w(\mathcal{S})$ and not on the original measure function $m(\mathcal{S})$. As emphasized above, we do not assume that the integral $\int dw(s)$ is finite when extended over the whole s-space. We do not even require that this integral extended over every bounded region be finite.

As before, we employ partitions $\mathcal{P}$ of the s-space into bounded and semi-infinite rectangular cells. The collection of all those cells $\mathcal{C}_k$ of $\mathcal{P}$ for which the weight

$$w_k = \int_{\mathcal{C}_k} dw(s) \tag{19.17}$$

is finite will be denoted by $\mathcal{R}$, the remainder by $\mathcal{C}_*$.

In contrast to our stipulations in Section 18 we do not require that the occupation $\nu(s)$ lead to a finite total number $\sum_s \nu(s)$; but we do require the total number

$$\nu_{\mathcal{C}} = \sum_{s \text{ in } \mathcal{C}} \nu(s) \tag{19.18}$$

to be finite for every bounded cell for which the weight $W(\mathcal{C})$ is finite. In view of the fact that the total number of particles is expected to be infinite if the field is myriotic, it would be incongruous—though not impossible— to restrict the occupations $\nu(s)$ to functions with $\sum_s \nu(s) < \infty$.

With reference to a partition $\mathcal{P}$ we again set

$$\nu_k = \sum_{s \text{ in } \mathcal{C}_k} \nu(s), \qquad \nu_* = \sum_{s \text{ in } \mathcal{C}_*} \nu(s). \tag{19.19}$$

According to our assumption, the numbers ν_k are required to be finite, but not the numbers ν_*.

In order to define the functionals $\phi(\nu)$, we introduce a notion of compatibility which differs from that introduced in Section 18. We say a functional $\tau(\nu)$ is *compatible in the second manner* with a partition $\mathcal{P}$ if it depends only on the values ν_k, but not on ν_*. We recall that in Section 17 a function $\tau(\nu)$ was called compatible—we would now say compatible in the first manner—with a partition $\mathcal{P}$ if it depends only on the values ν_k and ν_* (on ν_* in such a way as to vanish unless $\nu_* = 0$). For functionals $\tau(\nu)$ which are compatible with $\mathcal{P}$ in the second manner we introduce the "integral"

$$I_w(\tau) = \sum_{\nu_1, \nu_2, \cdots} \tau(\nu) \prod_{k \neq *} \nu_k ! w_k^{\nu_k} \exp\{-w_k\}.$$

It will be convenient to work with this expression; the inner product of two functionals will be derived from it as a special case.

Again it is obvious that the manifold of functionals which are compatible with some partition forms a linear space. It remains to show that the value of $I_w(\tau)$ is not changed when referred to a refinement $\mathcal{P}'$ of $\mathcal{P}$.

Denoting by $\mathcal{C}'_{rk}$ all cells of $\mathcal{P}'$ which compose the cell $\mathcal{C}_k$ of $\mathcal{P}$ we have

$$\sum_r \nu'_{rk} = \nu_k\,, \qquad \sum_r w'_{rk} = w_k$$

and hence

$$\sum_{\nu'_{1k},\nu'_{2k},\cdots} \prod_r \nu'_{rk}\text{¡}[w'_{rk}]^{\nu'_{rk}} \exp\{-w'_{rk}\} = \nu_k\text{¡}[w_k]^{\nu_k} \exp\{-w_k\}.$$

Therefore, the argument given in Section 18 applies to all cells $\mathcal{C}'$ of $\mathcal{P}'$ which lie in $\mathcal{R}$. A new argument is needed for those cells $\mathcal{C}'_{r*}$ of $\mathcal{P}'$ which lie in $\mathcal{R}'$ but outside of $\mathcal{R}$. Since the functional $\tau(\nu)$ is compatible with $\mathcal{P}$ in the second manner, its value does not depend on the values of ν'_{r*} assigned to these cells. Consequently, these cells contribute the factor

$$\sum_{\nu'_{1*},\nu'_{2*},\cdots} \prod_r \nu'_{r*}\text{¡}[w'_{r*}]^{\nu'_{r*}} \cdot \exp\{-w'_{r*}\}$$

to the expression $I_w(\tau)$, taken with reference to the refinement $\mathcal{P}'$. This factor has evidently the value 1. Thus, the invariance of the expression $I_w(\tau)$ under refinement is proved.

It is to be noted that both the presence of the exponential factor and the assumption that $\phi(\nu)$ does not depend on ν_*, are material for this invariance.

We say that a functional $\tau(\nu)$ "equals" zero if $I_w(|\tau|) = 0$. After having made this stipulation we can extend the linear space of compatible functionals $\tau(\nu)$ with respect to the norm $I_w(|\tau|)$ to a closed space $\mathfrak{L}_w$ by adjoining ideal elements. We then write symbolically

$$I_w(\tau) = \sum_\nu \tau(\nu) \prod_s \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\}. \tag{19.20}$$

In a similar way we construct a Hilbert space $\mathfrak{S}_w$. We begin with functionals $\phi(\nu)$ which are compatible with a partition $\mathcal{P}$ and define the unit form

$$(\phi, \phi)_w = \sum_{\nu_1,\nu_2,\cdots} |\phi(\nu)|^2 \prod_{k\neq *} \nu_k\text{¡}w_k^{\nu_k} \exp\{-w_k\}. \tag{19.21}$$

Evidently, the inner product can be expressed with the aid of the expression I_w as $(\phi, \phi)_w = I_w(|\phi|^2)$ and, therefore, we can conclude that the expression $(\phi, \phi)_w$ is not changed when referred to a refinement of the partition $\mathcal{P}$. Again we set $\phi(\nu) = 0$ if $(\phi, \phi)_w = 0$. The linear space of the compatible functionals $\phi(\nu)$ with this stipulation can be extended to a complete space $\mathfrak{S}_w$ by adjoining ideal elements. For the unit form we use the notation

$$(19.22) \qquad (\phi, \phi)_w = \sum_{\nu} | \phi(\nu) |^2 \prod_s \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\}$$

$$= I_w(| \phi |^2).$$

The inner product of two functionals $\phi(\nu)$ and $\hat{\phi}(\nu)$ in $\mathfrak{G}_w$ will be written as

$$(19.22)' \qquad (\phi, \hat{\phi})_w = \sum_{\nu} \bar{\phi}(\nu)\hat{\phi}(\nu) \prod_s \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\}.$$

It could easily be shown that to any pair of functionals $\phi(\nu)$, $\hat{\phi}(\nu)$ in $\mathfrak{G}_w$ there corresponds a functional $\tau(\nu)$ in $\mathfrak{L}_w$, denoted by $\bar{\phi}(\nu)\hat{\phi}(\nu)$ such that

$$(19.23) \qquad (\phi, \hat{\phi})_w = I_w(\bar{\phi}\hat{\phi}).$$

Conditions on $w(\mathcal{S})$ and $v(s)$

We assume that the weight function $w(\mathcal{S})$ can be generated from a function $v^-(s)$, the "weight amplitude", and its conjugate $v^+(s)$ such that relation

$$(19.16) \qquad dw(s) = | v(s) |^2 \, dm(s)$$

holds. We do not require that the weight $w(\mathcal{C}) = \int_{\mathcal{C}} dw(s)$ is finite for every bounded cell $\mathcal{C}$; we do require, however, that partitions $\mathcal{P}$ can be found such that the collection $\sum \mathcal{C}$ of cells $\mathcal{C}$ of this partition over which $w(\mathcal{C})$ is not defined has an arbitrarily small measure $\sum m(\mathcal{C})$. In addition, we require that to every function $\gamma^-(s)$ for which $\int | \gamma(s) |^2 \, dm(s) < \infty$ a function $\zeta(s)$ exists such that $\gamma^-(s) = \zeta(s)v^-(s)$. This condition is satisfied, for example, if $| v(s) | > 0$ throughout.

Annihilation and Creation Operators

With the aid of the functions $v^{\mp}(s)$ we shall define annihilation and creation operators acting on functionals $\phi(\nu)$ in $\mathfrak{G}_w$. Without being perfectly consistent in our notation we denote these operators by $A^{\mp}_{\nu}(s)$. We shall define these operators in such a way that they can be described symbolically by

$$(19.24)^- \qquad A^-_{\nu}(s')\phi(\nu) = v^-(s')\phi(\nu + \delta_{s'}),$$

$$(19.24)^+ \qquad A^+_{\nu}(s')\phi(\nu) = v^+(s')[dw(s')]^{-1}\nu(s')\phi(\nu - \delta_{s'}).$$

In a formal manner one verifies that these formulas are consistent with the formulas (19.4), when the functionals $\tilde{f}(\nu)$ and $\phi(\nu)$ are connected by (19.15). A formal verification can also be given for the commutation law

$$(19.25) \qquad [A^-_{\nu}(s'), A^+_{\nu}(s'')] = \delta_{s'}(s'')[dm(s')]^{-1} = \delta(s' - s'').$$

Note that Dirac's delta function $\delta(s' - s'')$ occurring on the right refers to the measure function $m(\mathcal{S})$ although $w(\mathcal{S})$ enters the right member of relation $(19.24)^+$.

Formulas (19.24) are the symbolic expression of relations

$$(19.26)^- \qquad \int \gamma^+(s)A^-_\nu(s)\, dm(s)\phi(\nu) = \int \zeta(s')\phi(\nu + \delta_{s'})\, dw(s'),$$

$$(19.26)^+ \qquad \int \gamma^-(s)A^+_\nu(s)\, dm(s)\phi(\nu) = \sum_{s'} \zeta(s')\nu(s')\phi(\nu - \delta_{s'}),$$

where $\gamma^{\mp}(s) = \zeta(s)v^{\mp}(s)$, as before; relation (19.25) expresses the statement that the commutator of the operators in (19.26) is

$$(19.27) \qquad \begin{aligned} &\left[\int \zeta_1(s)v^+(s)A^-_\nu(s)\, dm(s), \qquad \int \zeta_2(s)v^-(s)A^+_\nu(s)\, dm(s)\right] \\ &\qquad = \int |\, \zeta(s)\, |^2\, dw(s) = \int |\, \gamma(s)\, |^2\, dm(s). \end{aligned}$$

The operators $\int \zeta(s)v^{\mp}(s)A^{\pm}(s)\ dm(s)$ will, of course, first be defined for functionals $\phi(\nu)$ which are compatible with a partition in the second manner. In addition, we shall assume initially the function $\zeta(s)$ to be piecewise constant with respect to a partition; this implies that $\zeta(s) = 0$ on the cells $\mathfrak{C}$ for which $w(\mathfrak{C})$ is not defined. We may refer ϕ and ζ to the same common refinement $\mathcal{P}$ of the partitions to which they were referred originally. Setting

$$\zeta(s) = \zeta_k \qquad \text{in } \mathfrak{C}_k \qquad w(\mathfrak{C}_k) = W_k\,,$$

and

$$\nu_k = \sum_{s \text{ in } \mathfrak{C}_k} \nu(s),$$

as before, and

$$\phi(\nu) = \phi^{\mathcal{P}}(\nu_1\, .\, \nu_2\,, \cdots),$$

we define the right members of (19.26) as

$$\sum_k \zeta_k w_k \phi^{\mathcal{P}}(\nu_1 + \delta_1(k), \nu_2 + \delta_2(k), \cdots),$$

$$\sum_k \zeta_k\, \nu_k \phi^{\mathcal{P}}(\nu_1 - \delta_1(k)), \nu_2 - \delta_2(k), \cdots).$$

Now an arbitrary function $\zeta(s)$ may be approximated by one which is piecewise constant and an arbitrary functional $\phi(\nu)$ by one which is compatible; the right members of (19.26) may then be defined by a limit process—provided the function $\zeta(s)$ satisfies the condition

$$(19.28) \qquad \int |\, \zeta(s)\, |\, dw(s) < \infty$$

and the functional $\phi(\nu)$ satisfies the condition

$$(19.29)\qquad \sum_{\nu}\Big(\sum_{s} \mid \zeta(s) \mid \nu(s)\Big) \mid \phi(\nu) \mid^{2} \prod \nu(s)!\,[dw(s)]^{\nu(s)} \exp\{-dw(s)\} < \infty .$$

We do not want to give details here.

We note that the class of functions $\zeta(s)$ for which the operators $\int \zeta(s)v^{\pm}(s)\cdot A^{\pm}(s)\, dm(s)$ are defined depends on the weight function $w(S)$. This remark should be added as a qualification to the statement made above that there are different, non-equivalent, realizations of the creation and annihilation operators, satisfying commutation law (19.25).

The Functionals $\tilde{f}(\nu)$

The annihilation and creation operators acting on the functionals $\phi(\nu)$ depend on the function $v^{\pm}(s)$, see (19.17), and thus do not have the desired form (19.4). We may attain this form by introducing the functionals

$$\tilde{f}(\nu) = \phi(\nu) \prod_{s} [v^{-}(s)]^{\nu(s)},$$

instead of $\phi(\nu)$, see (19.15). The factor $\prod_{s} [v^{-}(s)]^{\nu(s)}$ occurring here is defined for occupations $\nu(s)$ for which $\sum_{s} \nu(s)$ is finite and for those functions $v^{-}(s)$ which tend to 1 sufficiently strongly as $\mid s \mid \rightarrow \infty$. Otherwise, we are forced to introduce $f(\nu)$ as a symbolic entity and to define the terms entering formulas (19.4) by

$$\tilde{f}(\nu + \delta_{s'}) = \phi(\nu + \delta_{s'}) \prod_{s} [v^{-}(s)]^{\nu(s)+\delta_{s'}(s)}$$

$$= \phi(\nu + \delta_{s'})v^{-}(s') \prod_{s} [v^{-}(s)]^{\nu(s)},$$

$$[dm(s')]^{-1}\nu(s')\tilde{f}(\nu - \delta_{s'}) = v^{-}(s')v^{+}(s')[dw(s')]^{-1}\nu(s')\phi(\nu - \delta_{s'}) \prod_{s} [v^{-}(s)]^{\nu(s)-\delta_{s'}(s)}$$

$$= [dw(s')]^{-1}v^{+}(s')\nu(s')\phi(\nu - \delta_{s'}).$$

The identity

$$(19.30)\qquad (\tilde{f}, \tilde{f})^{w} = (\phi, \phi)_{w} ,$$

(cf. (19.3) and (19.21)), which would result by formal arguments,

$$(\tilde{f}, \tilde{f})^{w} = \sum_{\nu} \mid f(\nu) \mid^{2} \prod_{s} \nu(s)!\,[dm(s)]^{\nu(s)} \exp\{-dw(s)\}$$

$$= \sum_{\nu} \mid \phi(\nu) \mid^{2} \prod_{s} \mid v(s) \mid^{2\nu(s)} \prod_{s} \nu(s)!\,[dm(s)]^{\nu(s)} \exp\{-dw(s)\}$$

$$= \sum_{\nu} \mid \phi(\nu) \mid^{2} \prod_{s} \nu(s)!\,[dw(s)]^{\nu(s)} \exp\{-dw(s)\}$$

$$= (\phi, \phi)_{w} ,$$

must also be adopted by definition.

Later on we shall employ only the functionals $\phi(\nu)$ instead of the functionals $\bar{f}(\nu)$ as representers of states; for this reason one need not worry about the difficulty of defining the functionals $\bar{f}(\nu)$ precisely.

We had referred to the functionals $\bar{f}(\nu)$ instead of the functionals $\phi(\nu)$ at the beginning of this section because we wanted to make it clear that the essential difference between the functionals introduced in this section and those introduced in Section 18 for ordinary fields lies in the presence of the factor $\exp\{-dw(s)\}$ in the unit form. The presence of the factor $\prod_s [v^-(s)]^{\nu(s)}$ is of secondary significance. In fact, even if $v^-(s) \equiv 1$, there would be a difference between the functionals $\phi(\nu)$ and $f(\nu)$. If $\int dm(s) = \infty$, the functionals $\phi(\nu)$ would represent states of a myriotic field, while the functionals $f(\nu)$ would represent the states of an ordinary field. Furthermore, it seems likely that various other functionals could take the place of the factor $\prod_s [v^-(s)]^{\nu(s)}$ provided the form (19.24) of the annihilation and creation operators is properly modified.

In case the total weight $W = \int dw(s)$ is finite, the functionals $\bar{f}(\nu)$ could be given a proper meaning by identifying them with the functionals $f(\nu) \exp\{\frac{1}{2}W\}$ of the first type. We may express this identification also by the formula

$$f(\nu) = \phi(\nu) \prod_s [v^-(s)]^{\nu(s)} \exp\{-\tfrac{1}{2}W\}. \tag{19.31}$$

Since the functionals $f(\nu)$ of the first type are defined only for occupations $\nu(s)$ with a finite total number $\sum_s \nu(s)$, the product $\prod_s [v^-(s)]^{\nu(s)}$ involves only a finite number of factors and may, therefore, be understood in the literal sense. Nevertheless, in order to establish the identification (19.31) properly, some lengthy—though standard—arguments are needed, which we do not want to give in detail. After having established this identification one is permitted to identify the terms ordinary and amyriotic.

Representation by Functionals $\phi(\nu)$

We shall find it convenient to use the functionals $\phi(\nu)$ in $\mathfrak{S}_\omega$ as representers of states Φ instead of the functionals $\bar{f}(\nu)$. We call these functions $\phi(\nu)$ "reduced" representers and write

$$\Phi \leftrightarrow \phi(\nu). \tag{19.32}$$

From formulas (19.24) we then deduce that the biquantized operator

$$A^+\lambda(S)A^- = \int A^+(s)\lambda(s)A^-(s)\, dm(s) \tag{19.33}$$

is represented by multiplication by

$$A^+_*\lambda(S)A^-_* = \sum_s \lambda(s)\nu(s). \tag{19.33$_*$}$$

Comparing this formula with (19.5) one sees that it does not matter whether the functionals $\phi(\nu)$ or $\bar{f}(\nu)$ are taken as representers. Equation $(19.33)_*$ shows that the operator $N(\mathfrak{C})$, corresponding to the number of particles in the cell $\mathfrak{C}$, is represented by

$$N_r(\mathfrak{C}) = \sum_{s \text{ in } \mathfrak{C}} \nu(s) = \nu(\mathfrak{C}) \tag{19.34}$$

as before, see (19.7) and (19.18).

Proof that Myriotic Fields Possess No Vacuum States

We proceed to prove the most striking property of myriotic fields, namely that they do not possess a vacuum state. The vacuum state Φ_{vac} may be defined by the property that for each cell $\mathfrak{C}$ the operator

$$N(\mathfrak{C}) = \int_{\mathfrak{C}} A^+(s)A^-(s)\, dm(s)$$

is applicable to the state Φ_{vac} and transforms it into $\Phi = 0$. We also may say that for each cell $\mathfrak{C}$ the expectation value $\langle N(\mathfrak{C})\rangle_{\Phi_{\text{vac}}}$ of the number $N(\mathfrak{C})$ of particles in the cell $\mathfrak{C}$ is defined for Φ_{vac} and has the value zero. The reduced representer $\phi_{\text{vac}}(\nu)$ of such a state is obviously characterized by the condition

$$\sum_{\nu} \nu(\mathfrak{C}) \mid \phi_{\text{vac}}(\nu) \mid^2 \prod_{s} \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\} = 0 \tag{19.35}$$

for every cell $\mathfrak{C}$. Here, as before, $\nu(\mathfrak{C}) = \sum_{s \text{ in } \mathfrak{C}} \nu(s)$. The statement that no vacuum state exists if the field is myriotic then reduces to the statement $\phi_{\text{vac}}(\nu) \equiv 0$ if $W = \infty$.

Since the functional $\phi_{\text{vac}}(\nu)$ may be an ideal element we really should prove this: *Suppose $\phi^\tau(\nu)$ is a sequence of compatible functionals for which the relations*

$$\begin{aligned}(\phi^\sigma - \phi^\tau, \phi^\sigma - \phi^\tau)_w \\ = \sum_{\nu} \mid \phi^\sigma(\nu) - \phi^\tau(\nu) \mid^2 \prod_{s} \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ \to 0 \quad \text{as} \quad \sigma, \tau \to \infty\end{aligned} \tag{19.36}$$

and

$$\begin{aligned}(\phi^\tau, N_r(\mathfrak{C})\phi^\tau)_w = \sum_{\nu} \nu(\mathfrak{C}) \mid \phi^\tau(\nu) \mid^2 \prod_{s} \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ \to 0 \quad \text{as} \quad \tau \to \infty\end{aligned} \tag{19.37}$$

hold; then also relation

$$\begin{aligned}(\phi^\tau, \phi^\tau)_w = \sum_{\nu} \mid \phi^\tau(\nu) \mid^2 \prod_{s} \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ \to 0 \quad \text{as} \quad \tau \to \infty\end{aligned} \tag{19.38}$$

holds.

Relation (19.36) means that to every $\epsilon > 0$ a $\sigma_\epsilon > 0$ exists such that

$$(19.39) \qquad (\phi^\sigma - \phi^\tau, \phi^\sigma - \phi^\tau)_w \le \epsilon \qquad \text{if} \qquad \sigma \ge \sigma_\epsilon \,, \tau \ge \tau_\epsilon \,.$$

From now on we shall choose $\sigma = \sigma_\epsilon$. Furthermore, we shall choose the cell $\mathcal{C} = \mathcal{C}_\epsilon$ in such a way that it contains all bounded cells of the partition $\mathcal{P}$ with which the functional $\phi^\sigma(\nu) = \phi^{\sigma_\epsilon}(\nu)$ is compatible, and that $w(\mathcal{C}) = \int_{\mathcal{C}} dw(s)$ is so large that

$$| \phi^\sigma(0) |^2 \exp \{-w(\mathcal{C})\} \le \epsilon.$$

Here $\phi^\sigma(0)$ is the value of $\phi^\sigma(\nu)$ for $\nu(s) \equiv 0$. Such a cell $\mathcal{C}$ can be found since, by assumption, $\int dw(s) = \infty$.

We assume that the functionals which will occur are compatible with partitions $\mathcal{P}$ so chosen that the cell $\mathcal{C}$ is composed of a finite number of cells of $\mathcal{P}$; such a partition can always be found by proper subdivision. We introduce two linear subspaces in the space of all such compatible functionals $\phi(\nu)$. The functionals $\phi_1(\nu)$ are characterized by the condition $\phi_1(\nu) = 0$ for all $\nu(s)$ with $\nu_{\mathcal{C}} = 0$; the functionals $\phi_2(\nu)$ by the condition $\phi_2(\nu) = 0$ for all $\nu(s)$ with $\nu(\mathcal{C}) \ge 1$. Evidently the product $\phi_1(\nu)\phi_2(\nu)$ vanishes for any pair of such functionals. It is also clear that each compatible functional is the sum

$$\phi(\nu) = \phi_1(\nu) + \phi_2(\nu)$$

of two functionals of these two types.

Since the cell $\mathcal{C}$ contains all bounded cells of the partition $\mathcal{P}$, the value of $\phi^\sigma(\nu)$ is independent of the values of $\nu(s)$ for points outside of $\mathcal{C}$; the same is true of $\phi_2^\sigma(\nu)$. Now, $\phi_2^\sigma(\nu)$ is zero unless $\nu(s) \equiv 0$ in $\mathcal{C}$. In other words,

$$\phi_2^\sigma(\nu) = \begin{cases} \phi^\sigma(0) & \text{if} \quad \nu(s) \equiv 0 \text{ in } \mathcal{C}, \\ 0, & \qquad \nu(\mathcal{C}) \ge 1. \end{cases}$$

As a consequence, only the terms with $\nu(s) \equiv 0$ contribute to the value of $(\phi_2^\sigma \,, \phi_2^\sigma)_w$; hence

$$(\phi_2^\sigma \,, \phi_2^\sigma)_w = | \phi^\sigma(0) |^2 \exp \{-w(\mathcal{C})\} \le \epsilon.$$

Relation (19.38) implies

$$(\phi_2^\sigma - \phi_2^\tau \,, \phi_2^\sigma - \phi_2^\tau)_w \le (\phi^\sigma - \phi^\tau, \phi^\sigma - \phi^\tau)_w \le \epsilon;$$

therefore,

$$(\phi_2^\tau \,, \phi_2^\tau)_w \le 2(\phi_2^\sigma \,, \phi_2^\sigma)_w + 2\epsilon \le 4\epsilon \qquad \text{for every} \qquad \tau \ge \sigma = \sigma_\epsilon \,.$$

We estimate $(\phi_1^\tau \,, \phi_1^\tau)_w$ with the aid of relation (19.37). According to this relation there is a number $\tau_\epsilon \ge \sigma = \sigma_\epsilon$ such that

$$(\phi^\tau, N_s(\mathcal{C})\phi^\tau)_w \le \epsilon \qquad \text{for} \qquad \tau \ge \tau_\epsilon \,.$$

Consequently,

$$(\phi_1^\tau, N_v(\mathfrak{C})\phi_1^\tau)_w \leq \epsilon \qquad \text{for} \qquad \tau \geq \tau_\epsilon .$$

Now $\nu(\mathfrak{C}) \geq 1$ whenever $\phi_1^\tau(\nu) \neq 0$. Therefore,

$$(\phi_1^\tau, \phi_1^\tau)_w \leq (\phi_1^\tau, N_v(\mathfrak{C})\phi_1^\tau)_w \leq \epsilon \qquad \text{for} \qquad \tau \geq \tau_\epsilon .$$

Altogether

$$(\phi^\tau, \phi^\tau)_w = (\Phi_1^\tau, \Phi_1^\tau)_w + (\Phi_2^\tau, \Phi_2^\tau)_w \leq 5\epsilon \qquad \text{for} \qquad \tau \geq \tau_\epsilon ;$$

thus we have derived the statement to be proved.

By similar arguments one can prove the statement made earlier that *the number operator N cannot be defined for myriotic fields.* More precisely, for every state Φ of a myriotic field the expected value $\langle N(\mathfrak{C})\rangle_\Phi$ of the number of particles contained in a cell $\mathfrak{C}$ tends to infinity as the cell $\mathfrak{C}$ tends to cover the whole s-space. We do not give details of the proof.

Equidistribution State

While myriotic fields do not possess a vacuum state, they possess another state which plays a basic role similar to the role the vacuum state plays for ordinary fields. This is the state Φ_v with the reduced representer $\phi(\nu) = 1$,

(19.40) $$\Phi_v \overset{v}{\leftrightarrow} 1,$$

or

(19.40)′ $$\Phi_v \underset{\nu}{\leftrightarrow} \prod_s [v^-(s)]^{\nu(s)},$$

see (19.32) and (19.8). In case the total weight W is finite and hence the field ordinary, this state possesses the representer

$$\tilde{f}_v(\nu) = \prod_s [v^-(s)]^{\nu(s)} \exp\{-\tfrac{1}{2}W\}$$

of the first type as seen from (19.1) and (19.15). Hence the state Φ_v is an "equidistribution" state in the sense explained at the end of Section 18. We adopt the same designation also for the state Φ_v in case the field is myriotic.

In an improper sense we may say that the probability that the occupation distribution $N(s)$ is given by the function $\nu(s)$ when the field is in the state Φ_v is given by

$$\prod_s \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\},$$

and hence distributed according to Poisson's law. If we restrict ourselves to occupations $\nu(s)$ which assume only the values $\nu = 0$ or $\nu = 1$ we may say that the probability of an occupation $\nu(s)$ which assumes only the values $\nu = 0$ and $\nu = 1$ is proportional to the product

$$\prod_{\nu(s)=1} dw(s)$$

of the weighted measures of the places at which particles occur. Relative to these weights, given by the differential $dw(s)$, one may therefore say that in the state Φ_ν all places s are equally likely to be occupied. Therefore, the term "state of relative equidistribution" or simply "equidistribution state" may be justified. In a different way, this term can be motivated after having calculated the expected number of particles in any cell.

In order to illustrate the nature of the equidistribution state, we determine the probability that no particle will be found in a region $\mathcal{R}$ when the field is in this state. Assuming that $\mathcal{R}$ is the finite part of a partition $\mathcal{P}$, we obtain this probability from formula (19.21) by inserting the functional $\phi(\nu)$ which equals one for $\nu_1 = \nu_2 = \cdots = 0$, and zero otherwise. The result is

$$\Pr\,(N(\mathcal{R}) = 0) = \exp\,\{-W(\mathcal{R})\}. \tag{19.41}$$

Of course, this formula holds also for more general regions $\mathcal{R}$. Letting $\mathcal{R}$ expand so as to cover the whole space we find

$$\Pr\,(N = 0) = \exp\,\{-W\}. \tag*{(19.41)$_\infty$}$$

This formula shows that *in the equidistribution state of a myriotic field the probability is zero that no particle will be found, so that with certainty at least one particle will be found.* This fact confirms the statement that a myriotic field cannot be in a vacuum state.

Expectation Values

The expected value of the biquantized observable $\sum_s \lambda(s)N(s)$ in the equidistribution state can be calculated by formal operations which may be justified in the way described in Section 18. We find

$$\begin{aligned}
&\langle \sum_s \lambda(s)N(s)\rangle_{\Phi_\nu} \\
&\qquad = \sum_\nu \Big(\sum_{s'} \lambda(s')\nu(s')\Big) \prod_s \nu(s)\text{¡}\,[dw(s)]^{\nu(s)} \exp\,\{-dw(s)\} \\
&\qquad = \sum_\nu \Big(\sum_{s'} \lambda(s')\,dw(s')\Big) \prod_s (\nu(s) - \delta_{s'}(s))\text{¡}\,[dw(s)]^{\nu(s)-\delta_{s'}(s)} \exp\,\{-dw(s)\},
\end{aligned}$$

whence

$$\langle \sum_s \lambda(s)N(s)\rangle_{\Phi_\nu} = \int \lambda(s)\,dw(s) \tag{19.42}$$

or, symbolically,

$$\langle N(s)\rangle_{\Phi_\nu} = dw(s). \tag*{(19.42)$'$}$$

Note that $N(s)$ itself is not a proper operator while $\sum_s \lambda(s)N(s)$ is proper if $\int |\lambda(s)|\,dw(s) < \infty$.

The formulas (19.42), (19.42)′ are the analogues of formulas (18.37) and (18.39). Formula (19.43)′ may be interpreted by saying that the number of particles is expected to be equally distributed over the s-space relative to the weight $dw(s)$. Thus, as in Section 18, the term equidistribution state might be justified.

Infrared Catastrophe

The equidistribution state may be interpreted as the vacuum state with respect to the modified annihilation and creation operators

$$B^{\pm}(s) = A^{\pm}(s) - v^{\pm}(s). \tag{19.43}$$

For, applying this operator to the equidistribution state with the representer $\phi(\nu) \equiv 1$, see (19.40), we derive from (19.24)$^-$, with $\phi(\nu) = 0$, the relation

$$B^-(s')\Phi_e = 0 \tag{19.44}$$

for every s'. Thus, even if the field is myriotic with respect to the operators $A^{\pm}(s)$, it is ordinary with respect to the operators $B^{\pm}(s)$.

This fact has some bearing on the problem of the infrared catastrophe discussed in Part III, Section 13. We may consider the operators $B^{\pm}$ as the unmodified operators, denoted by $A^{\pm}$ in Part III, and the operators $A^{\pm}$ as the modified operators, setting $v^{\pm}(s) = q^{\pm}(s)$, see (13.18). The condition $\int | q(s) |^2 \cdot dm(s) = \infty$ by which the infrared catastrophe was characterized in Part III is then exactly the condition that the total weight W be infinite or, in other words, the condition that the field be myriotic with respect to the modified operators $A^{\pm}$. Since myriotic fields admit no particle representation with respect to the operators $A^{\pm}$, we were not able to treat problems such as the spectral representation of the modified energy $A^+\omega(S)A^-$ in Part III. Since we now know that an occupation number representation of the second type does exist for myriotic fields we are able to treat this problem. In fact, the modified energy operator $A^+\omega(S)A^- = \sum_s \omega(s)N(s)$ is simply represented by multiplication of the representer $\phi(\nu)$ by $\sum_s \omega(s)\nu(s)$. Since this factor is a function of the independent variable $\nu(s)$ it is clear that the representation of the states Φ by the functionals $\phi(\nu)$ gives a spectral representation of the energy operator $A^+\omega(S)A^-$. Various special questions connected with this spectral representation will be discussed in Section 20.

Occupation Functions of a Discrete Variable

It was stated earlier that myriotic and amyriotic fields can be associated with any quantum variable s, provided that s is not restricted to run over only a finite number of values. This statement implies that both kinds of fields can be associated with a discrete quantum variable s, which runs over a discrete set of eigenvalues $s = s^r$, $r = 1, 2, \cdots$. Thus, in particular, both kinds of fields are possible even if the fields are enclosed in a box. Naturally, this statement refers to the mathematical possibility of satisfying the basic equations set up to characterize the fields; no claim is made about the actual occurrence of both kinds of fields enclosed in a box.

We proceed to describe in detail the nature of the functionals $\phi(\nu)$ of the space $\mathfrak{S}_w$ which may represent states of myriotic or amyriotic fields with reference to a discrete quantum variable. As in Section 18, we assume that each bounded s-cell contains at most a finite number of eigenvalues s^r. The weight of any such cell $\mathfrak{C}$ is then the sum of the contributions $w_r = dw(s^r)$ of all eigenvalues s^r in $\mathfrak{C}$,

$$w(\mathfrak{C}) = \sum_{s^r \text{ in } \mathfrak{C}} w_r .$$

As in Section 18 we introduce the special occupation numbers

$$n_r = \nu(s^r)$$

and the special functionals

$$\phi(\nu) = \phi^0(n_1, n_2, \cdots).$$

In contrast to the stipulation concerning the functionals $f^0(n_1, n_2, \cdots)$ made there, we now stipulate that the functional $\phi^0(n_1, n_2, \cdots)$ depends only on a finite number of variables $n_1, n_2, \cdots$. For these functionals $\phi^0(n_1, n_2, \cdots)$ we define the unit form

$$(19.45) \qquad (\phi^0, \phi^0)_w = \sum_{n_1, n_2, \cdots} | \phi^0(n_1, n_2, \cdots) |^2 \prod_r n_r!w_r^{n_r} \exp \{-w_r\}.$$

In this summation the variables $n_1, n_2, \cdots, n_r, \cdots$ should be restricted to a finite number containing all those on which $\phi^0(n_1, n_2, \cdots)$ actually depends. Automatically there is no contribution from those variables n_r on which $\phi^0(n_1, n_2, \cdots)$ does not depend because of the relation

$$\sum_{n_r=0}^{\infty} n_r!w_r^{n_r} \exp \{-w_r\} = 1.$$

The linear space of functionals $\phi^0(n_1, n_2, \cdots)$ for which the form $(\phi^0, \phi^0)_w$ is finite will be denoted by $\mathfrak{S}_w^0$.

It is remarkable that this space $\mathfrak{S}_w^0$ is not complete. In other words, there are sequences of functionals $\phi_\sigma^0(\nu)$ for which

$$(\phi_\tau^0 - \phi_\sigma^0, \phi_\tau^0 - \phi_\sigma^0)_w \rightarrow 0 \qquad \text{as} \qquad \sigma, \tau \rightarrow \infty,$$

and which neither depend on $n_{\sigma+1}, n_{\sigma+2}, \cdots$ nor converge to a functional $\phi^0(n_1, n_2, \cdots)$ depending only on a finite number of the variables n_r. It would not be difficult to construct such a sequence. The reason for the incompleteness of the space $\mathfrak{S}_w^0$ is that in order to describe a functional $\phi^0(n_1, n_2, \cdots)$ its values must be given for all sequences $n_1, n_2, \cdots$; and these sequences form a non-denumerable set. Note that the reason for the completeness of the space $\mathfrak{B}^0$ of functionals $f^0(n_1, n_2, \cdots)$ of the first type, described in Section 18, was that the values of such a functional need be given only for those sequences $n_1, n_2, \cdots$ which have a finite number of terms different from zero; and the set of such sequences is denumerable.

The space $\mathfrak{S}_w^0$ can, of course, be closed off to a Hilbert space by introducing "ideal" functionals. This Hilbert space will be denoted by $\mathfrak{S}_w^0$. The ideal functionals will also be denoted by $\phi^0(n_1, n_2, \cdots)$ and the expression (19.45) will be adopted for the unit form.

It would be possible to identify the ideal functionals with proper functionals $\phi^0(n_1, n_2, \cdots)$ defined for almost all sequences $n_1, n_2, \cdots$. The qualification "almost" refers to a measure which, according to Kolmogorov [37], can be introduced in the manifold of all sequences $n_1, n_2, \cdots$. The functionals $|\phi^0(n_1, \cdots)|^2$ are then integrable with respect to this measure and the value of this integral is exactly the unit form (19.45). In fact, this unit form is nothing but a special case of the integral introduced by Kolmogorov. For our purposes it is not necessary to make use of these possibilities.

We must identify the space $\mathfrak{S}_w^0$ of functionals $\phi^0(n_1, n_2, \cdots)$ with the space $\mathfrak{S}_w$ of functionals $\phi(\nu)$ of the second type in the case, assumed here, that the quantum variable s has a discrete spectrum of values s^r, $r = 1, 2, \cdots$. To this end we may call an occupation $\nu(s)$ "effective" if it vanishes unless s is such an eigenvalue. Furthermore we call a functional $\phi(\nu)$ "effective" if it vanishes for all non-effective occupations and depends only on a finite number of variables $\nu(s^r) = n_r$; such an effective functional is evidently a functional $\phi^0(\nu)$ in $\mathfrak{S}_w^0$. It can then be shown that every such effective functional for which expression (19.45) is finite is a functional in $\mathfrak{S}_w$, i.e. the limit of a sequence of compatible functionals in $\mathfrak{S}_w$; one can also show that each compatible functional in $\mathfrak{S}_w$ "equals" such an effective functional. In this way one can establish the identity of the spaces $\mathfrak{S}_w^0$ and $\mathfrak{S}_w$.

Myriotic Field in a Box

Assume, for example, that the particles composing the field are confined to a box, so that the spectrum of the energy is discrete. We let $r = 1, 2, 3, \cdots$ be a label characterizing a complete set of orthogonal eigenstates for the energy with the eigenvalues w_r. To each such eigenstate we may assign the weight w_r. The states of the field may then be represented by functionals $\phi^0(n_1, n_2, \cdots)$ so that the probability of the occupation $n_1, n_2, \cdots$ is

$$|\phi^0(n_1, n_2, \cdots)|^2 \prod n_r ¡ w_r^{n_r} \exp\{-w_r\}.$$

The field is myriotic if the weights are so chosen that $\sum_r w_r = \infty$; that is in particular the case if all weights w_r are equal. Then the field can not be in an eigenstate for the occupation $N(s)$, i.e. in a state in which the probability that the occupation differs from a particular one is zero. Describing this statement in terms of oscillators assigned to the various quantum numbers r, we may say that a myriotic field cannot be in a state in which each oscillator is excited to a definite level n_r. Such an eigenstate of the occupation would be an eigenstate of the total energy $\sum_r \omega_r N_r$; it is thus indicated that a myriotic field cannot be in a stationary state in a strict sense.

On the other hand, we know that a myriotic field may in a state of "relative equidistribution" be characterized by the representer $\phi^0(n_1, n_2, \cdots) \equiv 1$. The probability

$$\prod_r n_r!\, w_r^{n_r} \exp\{-w_r\},$$

of the occupation $n_1, n_2, \cdots$ in this state is evidently distributed according to Poisson's law.

20. *Probabilities and Expectation Values for the Equidistribution State*

In this section we shall evaluate the inner product of states which can be constructed by various procedures from the equidistribution state Φ_w. In terms of the reduced representers $\phi(\nu)$, $\phi^{(1)}(\nu)$ of these states the inner product can be written in the form

$$I_w(\tau) = \sum_{\nu} \tau(\nu) \prod_s \nu(s)!\,[dw(s)]^{\nu(s)} \exp\{-dw(s)\} \tag{20.1}$$

with $\tau(\nu) = \bar{\phi}(\nu)\phi^{(1)}(\nu)$; see (19.22), (19.23). Therefore, we shall evaluate the expression $I_w(\tau)$ for various types of functionals $\tau(\nu)$.

First we shall show that the expression $I_w(\tau)$ is easily evaluated if $\tau(\nu)$ is a polynomial in ν. Since the inner product of states produced from Φ_w by applying a finite number of creation operators can be written as such an expression it will be easy to evaluate inner products of this kind.

Secondly, we shall evaluate the expression $I_w(\tau)$ for functions $\tau(\nu)$ of the form $\tau(\nu) = H(\sum_s \lambda(s)\nu(s))$ in which $H(\alpha)$ is any function which can be represented by a complex integral of a certain kind. Note that $\sum_s \lambda(s)\nu(s)$ is the representer of the observable $\sum_s \lambda(s)N(s)$ obtained through biquantization of the observable $\lambda(s)$. Hence $H(\sum_s \lambda(s)\nu(s))$ is the representer of the function $H(\sum_s \lambda(s)N(s))$ of $\sum_s \lambda(s)N(s)$.

Using an appropriate step function $H(\alpha)$ we shall be able to calculate the probability $P(\kappa)$ that the observable $\sum_s \lambda(s)N(s)$ has a value less than any given number κ when the field is in the state Φ_w.

Another question which we shall answer arises in the following situation. Suppose that the value of the observable $\sum_s \lambda(s)N(s)$ has been measured and that a value between κ and $\kappa + d\kappa$ has been found; then the field is in a new state, $\Phi_{w,\kappa}$. We then ask for the expected value $\langle N(s)\rangle_{\Phi_{w,\kappa}}$ of the occupation distribution $N(s)$ in this new state. The answer is contained in the remarkably simple formula

$$\langle N(s)\rangle_{\Phi_{w,\kappa}} = [P'(\kappa)]^{-1}P'(\kappa - \lambda(s))\, dw(s). \tag{20.2}$$

Under suitable circumstances this expression is approximately given by the expression

$$\langle N(s)\rangle_{\Phi_\nu,\kappa} = \exp\{-\theta_\kappa\lambda(s)\}\, dw(s) \tag{20.3}$$

with an appropriate number θ_κ depending on κ.

Calculations of this type are very familiar from the theory of statistical mechanics, cf. [38] and [39], but they occur now in an essentially different situation. There an ensemble of stationary states of the field is considered, each with a definite weight. Here we consider one state of the field which, in general, is not stationary. This state may, however, be considered as a linear combination of stationary states with coefficients whose squares take the place of weights. Consequently, there is a strict analogy between the probabilities and expectation values to be evaluated here and the corresponding probabilities and expectation values in statistical mechanics. Incidentally, if $\lambda(s) = \omega(s)$ is the energy, the quantity θ_κ^{-1} is the analogue of the temperature.

One of the reasons that we present these calculations is that in statistical mechanics the assumption is, in general, made that the manifold of stationary states is discrete. In our notation this would mean that the values of the quantum variable s are restricted to a discrete spectrum; we intend to show that such a restriction is not necessary.

Among the many questions that could be answered by explicit evaluation, we have selected some that do not involve the expectation value of the total number of particles and therefore may be asked also for myriotic fields. To answer these questions the occupation number representation seems the appropriate tool, while for ordinary fields the particle representation could be employed with somewhat greater ease.

The methods used for the calculations are closely related to, and to a large extent identical with, familiar methods used in the theory of probability. Nevertheless, these calculations are given here because of the specific significance of our results in field quantum theory. However, we do not intend to justify all evaluations with complete rigor.

Evaluation of $I_\nu(\tau)$ for Polynomials $\tau(\nu)$

We shall at first be concerned with the evaluation of the inner product of two states which can be derived from the equidistribution state Φ_ν by applying creation operators a finite number of times. Specifically, we form such new states from the equidistribution state Φ_ν by applying n-fold creation operators such as

$$\begin{aligned} &A_n^+\{\zeta_n\} \\ &\quad = \int \cdots \int A^+(s_1) \cdots A^+(s)_n \zeta_n(s)_n v^-(s_1) \cdots v^-(s)_n \, dm(s_1) \cdots dm(s_r), \end{aligned} \tag{20.4}$$

where $\zeta_n(s)_n$ is an appropriate symmetric function of $s_1, \cdots, s_n$. Since the reduced representer of the state Φ, is the functional $\phi(\nu) \equiv 1$, cf. (19.40), the reduced representer of the state $A_n^+\{\zeta_n\}\Phi$, is

$$\phi(\nu) = \sum_{s_1, \cdots, s_n} \zeta_n(s)_n \nu(s_1) \cdots \nu(s_n), \tag{20.5}$$

as seen from $(19.19)^+$. It is of some interest to know how to evaluate the unit form of such a state or, more generally, the inner product of two such states. Denoting by $\phi_1(\nu)$ and $\phi_2(\nu)$ the representers of two such states, the inner product is given by expression

$$I(\bar{\phi}_1\phi_2) = \sum_\nu \bar{\phi}_1(\nu)\phi_2(\nu) \prod_s \nu(s)¡[dw(s)]^{\nu(s)} \exp\{-dw(s)\}, \tag{20.6}$$

cf. (19.20), (19.22). Evidently, the product $\bar{\phi}_1(\nu)\phi_2(\nu)$ is a polynomial in $\nu(s)$ and the evaluation of $I(\bar{\phi}_1\phi_2)$ is therefore a special case of the evaluation of the expression $I(\tau)$ for polynomials $\tau(\nu)$.

The evaluation of the expression $I_w(\tau)$ is particularly simple for a special class of polynomials, which we term "pseudo-polynomials." First we assign to every power ν^σ of the integer ν the polynomial

$$[\nu^\sigma]_{ps} = \nu(\nu - 1) \cdots (\nu - \sigma + 1), \tag{20.7}$$

called the "pseudo-power". To the product $\nu(s_1)\nu(s_2) \cdots \nu(s_r)$ of the values of $\nu(s)$ at the points $s_1, \cdots, s_r$ we then assign the "pseudo-product" $[\nu(s_1) \cdots \nu(s_r)]_{ps}$ in the following manner. We denote by ρ the number of the distinct points $s^1, \cdots, s^\rho$ among the points $s_1, \cdots, s_r$, and by $\sigma_1, \cdots, \sigma_\rho$ the number of times each ρ occurs, so that $\sigma_1 + \cdots + \sigma_\rho = r$. Then we set

$$[\nu(s_1) \cdots \nu(s_r)]_{ps} = [\nu^{\sigma_1}(s^1)]_{ps} \cdots [\nu^{\sigma_\rho}(s^\rho)]_{ps} . \tag{20.8}$$

To any homogeneous polynomial

$$p_r(\nu) = \sum_{s_1, \cdots, s_r} a_r(s)_r \nu(s_1) \cdots \nu(s_r) \tag{20.9}$$

we finally assign the homogeneous "pseudo-polynomial"

$$[p_r(\nu)]_{ps} = \sum_{s_1, \cdots, s_r} a_r(s)_r [\nu(s_1) \cdots \nu(s_r)]_{ps} . \tag{20.10}$$

We now maintain that the value of the expression I_w for such a pseudo-polynomial is given by

$$I_w([p_r]_{ps}) = \int \cdots \int a_r(s)_r \, dw(s_1) \cdots dw(s_r), \tag{20.11}$$

provided

$$\int \cdots \int | a_r(s)_r | \, dw(s_1) \cdots dw(s_r) < \infty . \tag{20.12}$$

Stripping the coefficients a_r from relation (20.11), we may express this relation symbolically by the formula

$$I_w([\nu(s_1) \cdots \nu(s_r)]_{ps}) = dw(s_1) \cdots dw(s_r). \tag{20.13}$$

The evaluation of the expression $I_w(\tau)$ for a general polynomial $\tau(\nu)$ can easily be deduced from formula (20.11). To do this we need only write the given polynomial as a pseudo-polynomial which can be done in a routine fashion by making use of the recursion formula

$$[\nu(s_1) \cdots \nu(s_r)]_{ps}\nu(s) = [\nu(s_1) \cdots \nu(s_r)\{\nu(s) + \delta_{s_1}(s) + \cdots + \delta_{s_r}(s)\}]_{ps}\,, \tag{20.14}$$

which immediately follows from (20.8). Thus we find, for example,

$$\nu(s_1)\nu(s_2) = [\nu(s_1)\nu(s_2) + \nu(s_1)\delta_{s_1}(s_2)]_{ps}\,, \tag*{(20.14)$_1$}$$

$$\begin{aligned}\nu(s_1)\nu(s_2)\nu(s_3) = [&\nu(s_1)\nu(s_2)\nu(s_3) + \nu(s_1)\nu(s_3)\delta_{s_1}(s_2) + \nu(s_1)\nu(s_2)\delta_{s_2}(s_3) \\ &+ \nu(s_2)\nu(s_3)\delta_{s_2}(s_1) + \nu(s_1)\delta_{s_1}(s_2)\delta_{s_2}(s_3)]_{ps}\,.\end{aligned} \tag*{(20.14)$_2$}$$

We shall deduce formula (20.11), or rather (20.13), by formal operations. The validity of the result could be proved rigorously in the way outlined in Section 18, by first assuming $a_r(s)_r$ to be piecewise constant and then approximating a general function $a_r(s)_r$ by such piecewise constant functions.

The formal proof proceeds as follows: Using (20.8) we write

$$\begin{aligned}I_w(&[\nu(s_1) \cdots \nu(s_r)]_{ps}) \\ &= \sum_{\nu} [\nu^{\sigma_1}(s^1)]_{ps} \cdots [\nu^{\sigma_r}(s^r)]_{ps} \prod_{s} \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ &= \sum_{\nu} [\nu^{\sigma_1}(s^1)]_{ps} \cdots [\nu^{\sigma_r}(s^r)]_{ps} \prod_{s=s^1,\cdots,s^\rho} \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\}.\end{aligned}$$

Obviously,

$$[\nu^{\sigma_\rho}(s^\rho)]_{ps}\nu(s^\rho)\text{¡} = (\nu(s^\rho) - \sigma_\rho)\text{¡};$$

therefore, when $\nu(s^\rho)$ is replaced by $\nu(s^\rho) + \sigma_\rho$ for $\rho = 1, \cdots, r$, the $I_w([\nu(s_1) \cdots \nu(s_r)]_{ps})$ assumes the form

$$\begin{aligned}&\sum_{\nu} \nu(s^1)\text{¡}[dw(s^1)]^{\nu(s^1)+\sigma_1} \exp\{-dw(s^1)\} \cdots \nu(s^\rho)\text{¡}[dw(s^\rho)]^{\nu(s^\rho)+\sigma_\rho} \exp\{-dw(s^\rho)\} \\ &\qquad = \Big(\sum_{\nu(s^1)} \nu(s^1)\text{¡}[dw(s^1)]^{\nu(s^1)+\sigma_1} \exp\{-dw(s^1)\}\Big) \\ &\qquad\qquad \cdots \Big(\sum_{\nu(s^r)} \nu(s^r)\text{¡}[dw(s^r)]^{\nu(s^r)+\sigma_r} \exp\{-dw(s^r)\}\Big) \\ &\qquad = [dw(s^1)]^{\sigma_1} \cdots [dw(s^\rho)]^{\sigma_\mu} = dw(s_1) \cdots dw(s_r).\end{aligned}$$

Thus formula (20.13) is obtained.

Formula (20.11) will first be applied to a linear polynomial $p(\nu) =$

$\sum_s \zeta(s)\nu(s)$. This polynomial is obviously its own pseudo-polynomial. Therefore, (20.11) yields

$$I_w\left(\sum_s \zeta(s)\nu(s)\right) = \int \zeta(s)\, dw(s), \tag{20.15}$$

in agreement with (19.41). Next we consider the square $p(\nu) = \mid \sum_s \zeta(s)\nu(s) \mid^2$ of the polynomial $\sum_s \zeta(s)\nu(s)$. We easily derive from (20.14), the identity

$$\mid \sum_s \zeta(s)\nu(s) \mid^2 = \left[\sum_{s',s''} \bar{\zeta}(s')\zeta(s'')\nu(s')\nu(s'') + \sum_s \mid \zeta(s) \mid^2 \nu(s)\right]_{ps} ;$$

hence formula (20.11) yields

$$I_w\left(\mid \sum_s \zeta(s)\nu(s) \mid^2\right) = \left| \int \zeta(s)\, dw(s) \right|^2 + \int \mid \zeta(s) \mid^2 dw(s). \tag{20.16}$$

Note that the functional $\phi(\nu) = \sum_s \zeta(s)\nu(s)$ is the representer of the state

$$A_1^+\{\zeta\}\Phi_\nu = \int A^+(s)\zeta(s)v^-(s)\, dm(s)\Phi_\nu ; \tag{20.17}$$

therefore, formula (20.16) gives the value of the unit form $\| A_1^+\{\zeta\}\Phi_\nu \|^2$ of this state.

Using the process described by formula (20.14) one could easily calculate the inner products of states obtained by applying any number of creation operators on the equipartition state Φ_ν.

Evaluation of the Expression $I_w(\tau)$ by Complex Integration

We proceed to evaluate the expression $I_w(\tau)$ for a more general class of functionals $\tau(\nu)$, namely, for functionals of the form

$$\tau(\nu) = H\left(\sum_s \lambda(s)\nu(s)\right). \tag{20.18}$$

Here $\lambda(s)$ is a function of s with properties to be specified and $H(\alpha)$ is a function of one variable α. We recall that $H(\sum_s \lambda(s)\nu(s))$ is the representer of the observable $H(\sum_s \lambda(s)N(s))$. In particular, we are interested in step functions $H(\alpha) = 1$ for $\alpha < \kappa$, $H(\alpha) = 0$ for $\alpha \geq \kappa$; the corresponding operator is then the projection into states in which $\sum_s \lambda(s)N(s)$ with certainty has a value $< \kappa$.

If the function $H(\alpha)$ were a polynomial in α or could easily be approximated by polynomials, the results of the preceding subsection could be used. For step functions, and various other functions, a different approach is much more suitable. In this approach we assume that the function $H(\alpha)$ admits the representation

$$H(\alpha) = \int_{\mathfrak{L}} \exp\{z\alpha\}h(z)\, dz \tag{20.19}$$

formed with the aid of an analytic function $h(z)$ which is regular in the neighborhood of the path $\mathfrak{L}$ and behaves appropriately when z approaches infinity along the path.

The advantage of the representation (20.19) is that it enables one to reduce the evaluation of $I_w(H(\sum_s \lambda(s)\nu(s)))$ to the evaluation of $I_w \exp\{z \sum_s \lambda(s)\nu(s)\}$; for, this expression can be evaluated explicitly.

We proceed in a formal manner: We substitute $\sum_s \lambda(s)\nu(s)$ for α in (20.19) and form the expression $I_w(\tau)$ for the resulting functional. By virtue of the linear dependence of $I_w(\tau)$ on τ we obtain

$$I_w(H(\sum_s \lambda(s)\nu(s))) = \int_{\mathfrak{L}} I_w(\exp\{z \sum_s \lambda(s)\nu(s)\})h(z)\, dz. \tag{20.20}$$

Writing the exponential function occurring here as a product,

$$\begin{aligned}\exp\{z \sum_s \lambda(s)\nu(s)\} &= \prod_s \exp\{z\lambda(s)\nu(s)\} \\ &= \prod_s \exp\{z\lambda(s)\}^{\nu(s)}\end{aligned}$$

and using the formula

$$\sum_{\nu(s)} \nu(s)¡\, a^{\nu(s)} = \exp\{a\}$$

we obtain

$$\begin{aligned}I_w(\exp\{z \sum_s \lambda(s)\nu(s)\}) &= \sum_\nu \prod_s \exp\{z\lambda(s)\}^{\nu(s)} \prod_s \nu(s)¡\,[dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ &= \sum_\nu \prod_s \nu(s)¡\,[\exp\{z\lambda(s)\}\, dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ &= \prod_s \Big(\sum_{\nu(s)} \nu(s)¡\,[\exp\{z\lambda(s)\}\, dw(s)]^{\nu(s)}\Big) \exp\{-dw(s)\}.\end{aligned}$$

By virtue of the formula $\sum_{\nu(s)} \nu(s)¡\, a^{\nu(s)} = \exp\{a\}$, the last expression equals

$$\begin{aligned}\prod_s \exp\{\exp\{z\lambda(s)\}\, dw(s)\} \exp\{-dw(s)\} &= \prod_s \exp\{\exp\{z\lambda(s)\} - 1\}\, dw(s) \\ &= \exp\Big\{\int [\exp\{z\lambda(s)\} - 1]\, dw(s).\Big\}\end{aligned}$$

Introducing the function

$$Y(z) = Y(z;\lambda) = \int [\exp\{z\lambda(s)\} - 1]\, dw(s) \tag{20.21}$$

we can write the result in the form

$$I_w(\exp\{z \sum_s \lambda(s)\nu(s)\}) = \exp\{Y(z;\lambda)\}. \tag{20.22}$$

Insertion of (20.22) into (20.20) gives the *basic formula*

$$I_w(H(\sum_s \lambda(s)\nu(s))) = \int_{\mathfrak{L}} \exp\{Y(z)\}h(z)\, dz. \tag{20.23}$$

The argument given is well known from the theory of probability. In the language of this theory we may say: For each value of s one may consider $\nu(s)$ as a random variable having a Poisson probability distribution with the mean $dw(s)$. All these variables are independent of each other. The same is true of the variables $\exp\{z\lambda(s)\nu(s)\}$. Hence

$$I_w(\exp\{z\sum_s \lambda(s)\nu(s)\}) = \prod_s I_w(\exp\{z\lambda(s)\nu(s)\}).$$

Since the mean of

$$\exp\{z\lambda(s)\nu(s)\} = \exp\{z\lambda(s)\}^{\nu(s)}$$

is

$$\exp\{\exp\{z\lambda(s)\} - 1\}\, dw(s),$$

identity (20.23) results.

The function $Y(z; \lambda)$ which occurs in identity (20.23) is connected with the function

$$Z(z) = \int \exp\{z\lambda(s)\}\, dw(s), \tag{20.24}$$

by the relation

$$Z(z) = Y(z) \exp\{W\}. \tag{20.25}$$

Since the function $Z(z)$ corresponds to the "partition function" in statistical mechanics we shall call $Y(z)$ the "adjusted partition function". The function $Z(z)$ would occur in formula (20.23) instead of $Y(z)$ if the functional $I_w(\tau)$ were defined in the first manner, i.e., without the factors $\exp\{-dw(s)\}$. Clearly, in the case $W = \int dw(s) = \infty$, in which we are particularly interested, only one of the two integrals, $Z(z)$ or $Y(z)$, can exist. Conditions will be formulated under which $Y(z)$, rather than $Z(z)$, is defined.

Conditions on $\lambda(s)$, $w(\mathfrak{C})$, and $h(z)$

It is, of course, possible to establish the basic formula (20.23) rigorously, defining at the same time the meaning of the functional $H(\sum_s \lambda(s)\nu(s))$, provided the functions $\lambda(s)$, $w(\mathfrak{C})$, and $h(z)$ satisfy appropriate conditions. We shall formulate a few such conditions, but we shall not carry out all the details.

First we require that the function $\lambda(s)$ is non-negative,

$$\lambda(s) \geq 0, \tag{20.26}$$

and integrable with respect to the weight $w(\mathfrak{C})$, i.e.

$$\int \lambda(s)\, dw(s) < \infty. \tag{20.27}$$

The number

$$\kappa_0 = \int \lambda(s)\, dw(s), \tag{20.28}$$

as seen from (19.41), is the expectation value of the observable $\sum_s \lambda(s)N(s)$, cf. (20.35), in the equidistribution state Φ_e.

Conditions (20.26), (20.27) imply that the weight of the region $\lambda(s) \geq l$ is finite for every $l > 0$,

$$\int_{\lambda(s)\geq l} dw(s) < \infty, \qquad l > 0. \tag{20.29}$$

Consequently, if $W = \int dw(s)$ is infinite, this infinity must originate from the weight w at those places s at which $\lambda(s) = 0$.

Under conditions (20.26), (20.27) the adjusted partition function $Y(z; \lambda)$ and its derivative $Y_z(z)$ are defined in the left half-plane, $\mathcal{R}e\, z = x \leq 0$. Specifically, the estimates

$$\mathcal{R}e\, Y(z) \leq 0, \tag{20.30}$$

$$| Y(z) | \leq \kappa_0 | z |, \tag{20.31}$$

$$| Y_z(z) | \leq \kappa_0 \tag{20.32}$$

are easily derived from (20.26), (20.27), (20.28).

More detailed information about the behavior of $Y(z)$ as $| z | \to \infty$ when $\mathcal{R}e\, z \leq 0$ can be obtained if more specific requirements are imposed on $\lambda(s)$ and $w(\mathcal{C})$. In particular, we can formulate conditions which insure that $Y(z)$ behaves in such a way that

$$| \exp \{Y(z)\} | \leq \gamma | z |^{-\beta} \quad \text{for} \quad | z | \geq \rho, \quad \mathcal{R}e\, z \leq 0 \tag{20.33}$$

with appropriate numbers $\beta, \gamma, \rho > 0$. To formulate such conditions, we may introduce the function

$$p(l) = -\int_{\lambda(s)\geq l} dw(s) \tag{20.34}$$

and require the existence of a non-negative number β such that the inequality

$$\int_0^{l_0} | dp(l) - \beta l^{-1}\, dl | \leq m_0 \tag{20.35}$$

holds for appropriate positive numbers l_0 and m_0. With

$$\phi(z) = \int_0^{l_0} [1 - \exp \{zl\}]l^{-1}\, dl$$

we then find

$$| Y(z) + \beta\phi(z) | \leq 2m_0 + 2p(l_0)$$

and, since

$$| \phi(z) - \log(-z) | \leq m_1 \qquad \text{for} \qquad | z | \geq \rho, \qquad \Re z \leq 0,$$

with appropriate m_1 and $\rho > 0$, we obtain

$$| Y(z) + \alpha \log(-z) | \leq 2m_1 + 2p(l_0) + \alpha m_1$$

whence (20.33) follows with an appropriate number γ.

Conditions (20.26), (20.27), (20.32) are satisfied in the case of the infrared catastrophe treated in Part III provided the quantities s, λ and w are interpreted as follows: $s = k$, $\lambda = \omega = | k |$, $dw = | q |^2 dk = \frac{1}{2} | j(k) |^2 | k |^{-3} dk$, see (13.18) and (13.54 to 13.61).

Condition (20.27) becomes

$$\frac{1}{2} \int | j(k) |^2 | k |^{-2} dk < \infty, \tag{20.36}$$

cf. (13.50), (13.51). From (20.31) we find

$$p(l) = -\frac{1}{2} \int_{|k| \geq l} | j(k) |^2 | k |^{-3} dk,$$

whence $l\, dp(l)/dl \to 2\pi | j(0) |^2$. Consequently, condition (20.32) is satisfied with $\beta = 2\pi | j(0) |^2$ if $j(k)$ is sufficiently smooth at $k = 0$.

Conditions on the function $h(z)$ under which the integral $\int_{\mathfrak{L}} \exp \{ Y(z) \} h(z)\, dz$ is defined can now easily be given. Suppose the function $Y(z)$, in the left half-plane, dies out at infinity strongly enough, e.g., suppose it satisfies condition (20.33). Then it is sufficient that, in this half-plane, the function $h(z)$ vanish of first order at infinity. In this case the line $x = x_1 < 0$ may be taken as the path $\mathfrak{L}$. In the following section we shall confine ourselves to the consideration of the function $z^{-1} \exp \{ -\kappa z \}$ with $\kappa \geq 0$.

Evaluation of Probabilities

We shall employ formula (20.23) in order to derive an explicit expression for the probability Pr (κ) that the biquantized observable $[\lambda(s)]$, which we also denote by

$$\mathrm{K} = \sum_s \lambda(s) N(s), \tag{20.37}$$

lies below a value κ if the field is in the equidistribution state Φ, . If we choose $s = k$ and $\lambda = \omega$, the quantity K is exactly the energy of the field.

Let us denote by $\eta_\kappa(\alpha)$ the shifted unit step function,

$$\begin{aligned} \eta_\kappa(\alpha) &= 1 \qquad \text{for} \qquad \alpha < \kappa, \\ &= 0 \qquad \text{for} \qquad \alpha \geq \kappa, \end{aligned} \tag{20.38}$$

and form the operator $\eta_\kappa(\mathrm{K})$. This operator projects any state into a state in which a measurement of the quantity K with certainty would yield a value $< \kappa$. The probability Pr (κ) can therefore be written as the inner product

$$\text{Pr}\,(\kappa) = (\Phi_\nu\,,\ \eta_\kappa(\mathrm{K})\Phi_\nu) \tag{20.39}$$

of the state Φ_ν into the state $\eta_\kappa(\mathrm{K})\Phi_\nu$. Since the state Φ_ν is represented by the functional $\phi(\nu) = 1$ and the operator $\eta_\kappa(\mathrm{K})$ is represented by multiplication by the functional $\eta_\kappa(\sum_s \lambda(s)\nu(s))$, the state $\eta_\kappa(\mathrm{K})\Phi_\nu$ is represented by the functional $\phi(\nu) = \eta_\kappa(\sum_s \lambda(s)\nu(s))$. By virtue of (19.21), therefore, the probability Pr (κ) is given by

$$\begin{aligned}\text{Pr}\,(\kappa) &= \sum_\nu \eta_\kappa\Big(\sum_s \lambda(s)\nu(s)\Big) \prod_s \nu(s)\text{¡}[dw(s)]^{\nu(s)} \exp\{-dw(s)\}\\ &= I_w\Big(\eta_\kappa\Big(\sum_s \lambda(s)\nu(s)\Big)\Big).\end{aligned} \tag{20.40}$$

We now observe that the unit step function $\eta_\kappa(\alpha)$ admits a representation of the form (20.19), namely

$$\eta_\kappa(\alpha) = \frac{1}{2\pi i}\int_{\mathcal{L}} \exp\{(\alpha - \kappa)z\}z^{-1}\,dz, \qquad \text{for} \qquad \alpha \neq \kappa, \tag{20.41}$$

when we take as path $\mathcal{L}$ any straight line $z = x_1 + iy$ on the left half-plane $x_1 < 0$ traversed from $y = +\infty$ to $y = -\infty$. The function $\eta_\kappa(\alpha)$ is therefore of the form (20.19) with $h(z) = z^{-1}\exp\{-\kappa z\}$.

We assume that the function $\lambda(s)$ and the weight $w(\mathcal{C})$ satisfy the conditions formulated in the previous subsection. Condition (20.27), in particular, does not appear to be unduly restrictive. For, if this condition is not satisfied, i.e. if the expectation value κ_0 of the observable K is infinite, the probability that K lies below a finite value will, in general, be zero.[3]

Having adopted these conditions, formula (20.23) is applicable. Combined with (20.38) it yields the main result

$$\text{Pr}\,(\kappa) = \frac{1}{2\pi i}\int_{\mathcal{L}} \exp\{Y(z) - \kappa z\}z^{-1}\,dz. \tag{20.42}$$

It may seem desirable to verify the relations

$$\text{Pr}\,(0) = 0, \tag{20.43}$$

$$\text{Pr}\,(\kappa) \to 1 \qquad \text{as} \qquad \kappa \to \infty, \tag{20.44}$$

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[3] As an example consider a field enclosed in a box; let $s = 1, 2, 3, \cdots$ be a label of the eigenstates of the energy $\omega(s)$ of single particles which we take as a function $\lambda(s)$ and assume $\omega(s) \to \infty$ as $s \to \infty$. Further assume that each of these states has the same weight $w > 0$, so that $\int \omega(s)\,dw(s) = w\,\Sigma_s\,\omega(s) = \infty$. Then the unadjusted, but not the adjusted partition function is defined. Suppose now that the field is myriotic and in the equidistribution state associated with the weight $dw(s)$. Then the probability that the total energy of the field lies below a finite value is zero.

from formula (20.42). The first relation could be derived by deforming the path $\mathcal{L}$ towards $-\infty$ and making use of the behavior of $Y(z)$ at infinity, as expressed by (20.35).

In order to derive relation (20.44) one may first set

$$1 - \Pr(\kappa) = \frac{1}{2\pi i}\int_{\mathcal{L}} [1 - \exp\{Y(z)\}] \exp\{-\kappa z\} z^{-1}\, dz \tag{20.45}$$

and then deform the path $\mathcal{L}$ into the imaginary axis. On this axis the function $|\exp\{-\kappa z\}|$ is bounded independently of κ. After integration by parts one can then deduce from (20.30) and (20.33) that the integral (20.45) can be made arbitrarily small by making κ sufficiently large.

Saddle Point Method

Provided $\kappa < \kappa_0$, the integral (20.42) may be calculated approximately with the aid of the *saddle point method.* Since the function

$$Y_z(x) = \int \exp\{x\lambda(s)\}\lambda(s)\, dw(s)$$

varies monotonically from 0 to κ_0 when $z = x$ varies from $-\infty$ to 0 along the negative real axis, it assumes every real value $< \kappa_0$ once on this axis. The point $z = x = -\theta_\kappa$ at which Y_z assumes the value κ,

$$Y_z(-\theta_\kappa) = \int \exp\{-\theta_\kappa\lambda(s)\}\lambda(s)\, dw(s) = \kappa, \qquad \kappa < \kappa_0, \tag{20.46}$$

is the saddle point for the real part $\Re e\,\{Y(z) - \kappa z\}$. Placing the path $\mathcal{L}$ through this point, we derive in the well-known manner (e.g. [42]) the approximate expression

$$\Pr(\kappa) \approx [2\pi\theta_\kappa^2 Y_{zz}(-\theta_\kappa)]^{-1/2}, \tag{20.47}$$

which is valid for $\kappa < \kappa_0$ if the ratio $|Y_{zzz}^2(-\theta_\kappa)Y_{zz}^{-3}(-\theta_\kappa)|$ is sufficiently small.

For values $\kappa > \kappa_0$ one may use expression (20.45) for $1 - \Pr(\kappa)$ and try to deform the path $\mathcal{L}$ as far to the right as possible.

We do not intend to present further details concerning the nature of the function $\Pr(\kappa)$. The primary aim of our discussion was to show that, even for myriotic fields, significant quantities can, in principle, be calculated.

Conditioned Equidistribution State

The last quantity to be discussed in this section is closely analogous to a quantity of great importance in statistical mechanics.

Assume the field to be in the equidistribution state Φ_s; suppose we wish to measure the value of the observable $K = \sum_s \lambda(s)N(s)$. The probability that this value will be found to lie between the values of κ and κ_1 is obviously $\Pr(\kappa_1) - \Pr(\kappa)$.

Suppose now the indicated measurement has been made and the observable K has been found to have a value between κ and κ_1. Then the field is in a new state which we denote by Φ_{v,κ,κ_1} and term a "conditioned" equidistribution state. The representer of this state is the functional

$$\phi(\nu) = \eta_{\kappa,\kappa_1}(\sum_s \lambda(s)\nu(s))[\Pr(\kappa_1) - \Pr(\kappa)]^{-1/2}. \tag{20.46}$$

Here we have used the abbreviation

$$\eta_{\kappa,\kappa_1}(\alpha) = \eta_{\kappa_1}(\alpha) - \eta_\kappa(\alpha); \tag{20.47}$$

cf. (20.38). The denominator $[\Pr(\kappa_1) - \Pr(\kappa)]^{1/2}$ insures that $\| \Phi_{v,\kappa,\kappa_1} \| = 1$. We intend to calculate the expected value $\langle\Gamma\rangle_{\Phi_{v,\kappa,\kappa_1}}$ of the observable

$$\Gamma = \sum_s \zeta(s)N(s) \tag{20.48}$$

in the state Φ_{v,κ,κ_1}. Simple expressions can be derived for this expectation value.

Because of (20.46) the desired expectation value can be written as

$$\begin{aligned}\langle\Gamma\rangle_{\Phi_{v,\kappa,\kappa_1}} &= (\Phi_{v,\kappa,\kappa_1}, \Gamma\Phi_{v,\kappa,\kappa_1})\\ &= I_w(\sum_s \zeta(s)\nu(s)\eta_{\kappa,\kappa_1}(\sum_s \lambda(s)\nu(s)))[\Pr(\kappa_1) - \Pr(\kappa)]^{-1}\end{aligned}$$

or, with the abbreviation

$$I_w(\sum_s \zeta(s)\nu(s)\eta_\kappa(\sum_s \lambda(s)\nu(s))) = Q(\kappa; \zeta), \tag{20.49}$$

as

$$\langle\Gamma\rangle_{\Phi_{v,\kappa,\kappa_1}} = [Q(\kappa_1; \zeta) - Q(\kappa; \zeta)][\Pr(\kappa_1) - \Pr(\kappa)]^{-1}. \tag{20.50}$$

It will be shown that the remarkably concise formula

$$Q(\kappa; \zeta) = \int \Pr(\kappa - \lambda(s'))\zeta(s')\, dw(s') \tag{20.51}$$

holds for the quantity $Q(\kappa; \zeta)$. If $\Pr(\kappa)$ is expressed as the complex integral (20.42) and the two integrations are interchanged, this formula becomes

$$Q(\kappa; \zeta) = \frac{1}{2\pi i}\int_{\mathfrak{L}} \exp\{Y(z) - \kappa z\}\left[\int \exp\{z\lambda(s')\}\zeta(s')\, dw(s')\right]z^{-1}\, dz. \tag{20.52}$$

Formula (20.51) shows that it is sufficient to know the function $P(\kappa)$ in order to be able to calculate the expectation value of $\Gamma = \sum_s \lambda(s)N(s)$.

Formula (20.52) could be derived in the same way as formula (20.42). a more direct—though not rigorous—procedure is the following: We introduce the expression

$$Q(\kappa; \lambda) = I_w(\sum_s \lambda(s)\nu(s)\eta_\kappa(\sum_s \lambda(s)\nu(s))) \tag{20.53}$$

which is of the form (20.20) with $H(\alpha) = \alpha\eta_\kappa(\alpha)$ and $h(z) = \exp\{-\kappa z\}/2\pi i z^2$, provided we choose the path $\mathfrak{L}$ as in (20.42). Therefore,

$$Q(\kappa;\lambda) = \frac{1}{2\pi i}\int_{\mathfrak{L}} \exp\{Y(z) - \kappa z\} z^{-2}\, dz. \tag{20.54}$$

Now, as seen from (20.53) and (20.49), the expression $Q(\kappa;\zeta)$ is nothing but the first variation of the expression $Q(\kappa;\lambda)$ considered as a functional of $\lambda(s)$ with $\delta\lambda(s) = \zeta(s)$ since $\eta_\kappa(\alpha)$ is the derivative of $[\alpha - \kappa]\eta_\kappa(\alpha)$.

On the other hand we can calculate the first variation of $Q(\kappa;\lambda)$ from (20.54). As seen from (20.21), the first variation of $Y(z)$ is $z\int\exp\{z\lambda(s)\}\zeta(s)dw(s)$; hence the first variation of the right member of (20.54) is exactly the right member of (20.52). Thus the latter formula is verified.

An approximate expression for $Q(\kappa;\zeta)$ can be obtained from the *saddle point method* under the circumstances indicated in the previous subsection. Assuming that $\zeta(s')$ vanishes whenever $\lambda(s')$ is not small compared with κ, we may employ the same saddle point $z = -\theta_\kappa$ for the approximate evaluation of expression (20.52) that was employed for the evaluation of Pr (κ), see (20.45). The result is evidently

$$Q(\kappa;\zeta) \approx \int \exp\{-\theta_\kappa\lambda(s')\}\zeta(s')\, dw(s')\ \mathrm{Pr}\,(\kappa). \tag{20.55}$$

Furthermore, let us assume that the value $\kappa_1 + d\kappa$ lies so near to κ that we may use the saddle point $z = -\theta_\kappa$ even for the approximate evaluation of Pr (κ_1). Then the denominator on the right hand side of (20.50) can be cancelled and the simple formula

$$\begin{aligned}\langle\Gamma\rangle_{\Phi_\nu,\kappa,\kappa+d\kappa} &= \langle\sum_{s'}\zeta(s')N(s')\rangle_{\Phi_\nu,\kappa,\kappa+d\kappa}\\ &\approx \int \exp\{-\theta_\kappa\lambda(s')\}\zeta(s')\, dw(s')\end{aligned} \tag{20.56}$$

results. For $\zeta(s) = \lambda(s)$ this formula reduces by (20.46) to the formula

$$\langle\mathrm{K}\rangle_{\Phi_\nu,\kappa,\kappa+d\kappa} = \langle\sum_{s'}\lambda(s')N(s')\rangle_{\Phi_\nu,\kappa,\kappa+d\kappa} = \kappa, \tag{20.57}$$

as it should. Relation (20.56) may be expressed symbolically by formula

$$\langle N(s')\rangle_{\Phi_\nu,\kappa,\kappa+d\kappa} = \exp\{-\theta_\kappa\lambda(s')\}\, dw(s'), \tag{20.58}$$

which expresses the expected number of particles occupying the place s' on the shell $\sum_s \nu(s)\lambda(s) = \kappa$. This formula corresponds closely to an important formula in statistical mechanics. The saddle point method employed here, of course, corresponds to the method of Darwin and Fowler [39].

21. *Occupation Number Representation For Fermion Fields*

Occupation number representations of the types described in the previous sections for boson fields can also be developed for fermion fields. In fact the notions introduced for boson fields will be used for fermion fields also, but various new notions will be needed in addition. We shall first give a formal description of the occupation representation for fermions and later on indicate how the terms employed in it can be defined rigorously.

Representation of the First Type

The occupation representers $f(\nu)$ for boson fields were introduced by relation (18.3),

$$f(\nu) = [n!]^{1/2}\psi_n(s)_n ,$$

connecting them with the particle representers $\psi_n(s)_n$. The occupation $\nu(s)$ was connected with the set $(s)_n$ of n points $s_1 , \cdots , s_n$ through relation

$$\nu(s) = \delta_{s_1}(s) + \cdots + \delta_{s_n}(s). \tag{21.1}$$

Essentially the same connection will be maintained for fermion fields; of course, there must be deviations due to the antisymmetry of the fermion representers $\psi_n(s)_n$. These representers change sign when the arrangement $\{s_1 , \cdots , s_n\}$ undergoes an odd permutation while the occupation $\nu(s)$—and hence the value of any function $f(\nu)$ of it—is independent of this arrangement. Therefore, relation (18.3) is to be replaced by the relation

$$f(\nu) = \pm[n!]^{1/2}\psi_n(s)_n \tag{21.2}$$

in which the signature of the right member is to be assigned to the arrangement $\{s_1 , \cdots , s_n\}$ in a proper way. This will be done below. The antisymmetry of the function $\psi_n(s)_n$ also implies that this function vanishes whenever two of the points $s_1 , \cdots , s_n$ are the same. As seen from (21.1), this is the case exactly when $\nu(s) > 1$ for at least one point s. Thus it is clear that the occupation representer $f(\nu)$ vanishes unless the occupation number $\nu(s)$ satisfies the condition

$$\nu(s) = 0 \qquad \text{or} \qquad \nu(s) = 1 \qquad \text{for all values of } s. \tag{21.3}$$

Occupations $\nu(s)$ satisfying this condition will be called "F-admitted," or simply "admitted." Functionals $f(\nu)$ which vanish unless the occupation $\nu(s)$ is F-admitted will also be called "F-admitted."

With the aid of the functional $F(\nu)$ defined by

$$F(\nu)\begin{cases} = 1 & \text{if} \quad \nu(s) \text{ is admitted,} \\ = 0 & \text{if} \quad \nu(s) \text{ is not admitted,} \end{cases} \tag{21.4}$$

we can transform any functional $f(\nu)$ into the F-admitted functional $F(\nu)f(\nu)$. Evidently, every F-admitted functional $f(\nu)$ is reproduced by multiplication by $F(\nu)$,

$$F(\nu)f(\nu) = f(\nu) \qquad \text{if} \qquad f(\nu) \text{ is } F\text{-admitted}. \tag{21.5}$$

So far in the discussion the functionals $f(\nu)$ have been understood in the literal sense. Eventually, we must enlarge the space of admitted functionals to a closed space $\mathfrak{B}^F$. This could be done directly. We prefer, however, to introduce the space $\mathfrak{B}^F$ as a subspace of the Hilbert space $\mathfrak{B}$ of all—proper and ideal—occupation representers for a boson field. To this end we shall extend the operator which consists in multiplication of proper functionals $f(\nu)$ by $F(\nu)$ to an operator F applicable to all "functionals" of $\mathfrak{B}$. This operator F is evidently a projector cf. (19.41). The associated closed subspace, consisting of all "functionals" of the form $Ff(\nu)$, will be defined as the Hilbert space $\mathfrak{B}^F$.

Of course, expression (18.10) for the unit form (f, f) of a functional $f(\nu)$ in $\mathfrak{B}$ also applies to functionals in $\mathfrak{B}^F$. It is, however, convenient to adopt the simpler expression

$$(f, f) = \sum_{\nu=0}^{1} | f(\nu) |^2 \prod_{s} [dm(s)]^{\nu(s)} \tag{21.6}$$

for $f(\nu)$ in $\mathfrak{B}^F$, which is motivated by the fact that $\nu(s)!$ has the value one whenever $\nu(s) = 0$ or 1.

We proceed to describe in more detail the relationship between particle and occupation representers $\psi_n(s)_n$ and $f(\nu)$ referred to above.

First let us assume the function $\psi_n(s)_n$ to be given. Let $\nu(s)$ be an F-admitted occupation for which

$$n = \sum_{s} \nu(s) < \infty. \tag{21.7}$$

The finiteness of this sum implies that there is a finite number of points s for which $\nu(s) > 0$. The F-admission of $\nu(s)$ implies that there are exactly n distinct points $s = s_1, \cdots, s_n$ for which $\nu(s) = 1$. This set, $(s)_n = \{s_1, \cdots, s_n\}$, is then to be inserted in the right member of formula (21.2).

The sign $\pm$ occurring in this formula must be so determined that it changes when $(s)_n = \{s_1, \cdots, s_n\}$ undergoes an odd permutation. In order to give this sign a definite value we assign a "standard" order to every set of n distinct points s.

For example, we may employ a lexicographic order to this end. Let $s^{(1)}, s^{(2)}, \cdots, s^{(m)}$ be the coordinates of the point s. Under the following circumstances we set $s_1 < s_2$: if $| s_1^{(1)} | < | s_2^{(1)} |$, if $| s_1^{(1)} | = | s_2^{(1)} |$ and $s_1^{(1)} < s_2^{(1)}$, if $s_1^{(1)} = s_2^{(1)}$ and $| s_1^{(2)} | < | s_2^{(2)} |$ and so on. This ordering procedure is possible if the measure $m(\mathfrak{R})$ of every bounded region $\mathfrak{R}$ is finite. Otherwise the procedure must be slightly modified.

The permutation that must be applied to the arrangement $(s)_n = \{s_1, s_2, \cdots, s_n\}$ of n distinct points in order to produce the standard order will be denoted by $P(s)_n$. With the aid of the ordering function

$$\begin{aligned} \sigma_n(s)_n &= 1 \qquad \text{if} \quad P(s)_n \text{ is even,} \\ \sigma_n(s)_n &= -1 \qquad \text{if} \quad P(s)_n \text{ is odd,} \end{aligned} \tag{21.8}$$

we then define the functional $f(\nu)$ specifically by the formula

$$f(\nu) = [n!]^{1/2}\sigma_n(s)_n\psi_n(s)_n \,. \tag{21.9}$$

Since both $\sigma_n(s)_n$ and $\psi_n(s)_n$ are antisymmetric, the value of $f(\nu)$ is independent of the arrangement $s_1, \cdots, s_n$.

Vice versa, let a functional $f(\nu)$ in $\mathfrak{B}^F$ be given. Let $s_1, \cdots, s_n$ be any set of n points and define the occupation $\nu(s)$ by relation (21.1). Then the value $\psi_n(s)_n$ of the function ψ_n for the particular arguments $(s)_n = \{s_1, \cdots, s_n\}$ can be expressed as

$$\psi_n(s)_n = [n!]^{1/2}\sigma_n(s)_n f(\nu) \tag{21.10}$$

in terms of the value $f(\nu)$ of the functional f for the particular occupation $\nu(s)$, given by (21.1).

The state Φ which has the particle representer $\psi_n(s)_n$ can just as well be represented by the functional $f(\nu)$ given by (21.9). This representation will be denoted by

$$\Phi \underset{\nu}{\leftrightarrow} f(\nu). \tag{21.11}$$

The creation and annihilation operators $A^{\mp}(s)$ for fermion states satisfy the conditions

$$[A^-(s'), A^+(s'')]_+ = \delta(s' - s''), \tag{21.12}$$

$$[A^-(s'), A^-(s'')]_+ = [A^+(s'), A^+(s'')]_+ = 0, \tag*{(21.12)$^{\mp}$}$$

see (8.20). Here the notation

$$[A, B]_+ = AB + BA \tag{21.13}$$

for the "anticommutator" of two operators A and B has been used.

In order to describe the effect which these annihilation and creation operators have on the representing functionals $f(\nu)$, we must relate the ordering of the n numbers $s_1, \cdots, s_n$ to the orderings of the numbers $s_1, \cdots, s_{n-1}$, and $s_1, \cdots, s_n, s_{n+1}$. To this end we assign to every admitted occupation $\nu(s)$ and every point s' the number

$$\tau(\nu, s') = \sigma_n(s)_n\sigma_{n+1}(s)_n(s'), \tag{21.14}$$

in which $(s)_n$ is defined by (21.1). Note that the value of τ is independent of the arrangement of the points $s_1, \cdots, s_n$. With the aid of this number τ we may describe the annihilation and creation operators by

$$A^-(s')\Phi \underset{\nu}{\leftrightarrow} \tau(\nu, s')f(\nu + \delta_{s'}), \tag*{(21.15)$^-$}$$

$$A^+(s')\Phi \underset{\nu}{\leftrightarrow} [dm(s')]^{-1}\tau(\nu - \delta_{s'}, s')\nu(s')F(\nu)f(\nu - \delta_{s'}). \tag*{(21.15)$^+$}$$

If the factor $F(\nu)$ were omitted from the right member of $(21.15)^+$, it could happen that this member would be different from zero for an occupation $\nu(s)$ with $\nu(s') = 2$, which would not be admitted. Furthermore we note that $\nu(s) - \delta_{s'}(s)$ is not defined if $\nu(s') = 0$; therefore, we stipulate

$$(21.16) \qquad \nu(s')f(\nu - \delta_{s'}) = 0 \quad \text{if} \quad \nu(s') = 0.$$

The identity of the description $(21.15)^-$ of the annihilation operator with the description (8.2) in terms of the particle representation is immediately verified by using (21.10) and (21.9).

In order to identify $(21.15)^+$ with $(8.11)_{A*\nu}$, we assume the representer $f(\nu)$ of Φ to be given and determine from $(8.11)_{A*\nu}$ the representer $f'(\nu)$ of $A^+(s')\Phi$. To this end $f'(\nu)$ will be expressed in terms of the particle representer $\psi'_n(s)_n$ of $A^+(s')\Phi$; this function will be expressed in terms of the particle representer $\psi_{n-1}(s)_{n-1}$ of Φ, which in turn will be expressed in terms of $f(\nu)$. Thus $f'(\nu)$ will be expressed in terms of $f(\nu)$.

Let $\nu(s) = \delta_{s_1}(s) + \cdots + \delta_{s_n}(s)$ be an occupation. If $\nu(s') = 0$, also $f'(\nu) = 0$, as seen from $(8.11)_{A*\nu}$ the right member of $(21.15)^+$ vanishes by (21.16). Let then $\nu(s') = 1$. Clearly, s' is one of the points $s_1, \cdots, s_n$. We first assume $s' = s_n$ and find

$$\psi_{n-1}(s)_{n-1} = [(n-1)!]^{1/2}\sigma(s)_{n-1}f(\nu - \delta_{s'})$$

from (21.10). Then from $(8.11)_{A*\nu}$ we have

$$\psi'_n(s)_n = n^{1/2}n^{-1}\psi_{n-1}(s)_{n-1}\delta(s' - s_n),$$

or, since Dirac's delta function is connected with Kronecker's delta function by $\delta(s' - s_n) = \delta_{s'}(s_n)/dm(s')$, see (18.17),

$$\begin{aligned}\psi'_n(s)_n &= n^{-1/2}\psi_{n-1}(s)_{n-1}[dm(s')]^{-1}\\ &= [n!]^{1/2}\sigma(s)_{n-1}f(\nu - \delta_{s'})[dm(s')]^{-1},\end{aligned}$$

whence by (21.9)

$$f'(\nu) = \sigma(s)_{n-1}(s')\sigma(s)_{n-1}f(\nu - \delta_{s'})[dm(s')]^{-1}.$$

Because of (21.4) this is just the right member of $(21.15)^+$. Since this right member is independent of the position of s in the arrangement $s_1, \cdots, s_n$, the same result would have been found for any such position instead of $s' = s_n$. Finally, let $\nu(s') > 1$; then the right member of $(21.15)^+$ vanishes because of (21.4). Thus it is shown that the annihilation and creation operators possess the occupation representation (21.15) if they possess the particle representation (8.11).

Independently of this particle representation one could verify that operators $A^{\pm}(s)$ defined by the representation (21.15) satisfy the commutation laws (21.12).

The expressions (21.15) are the natural generalizations of the expressions given by Jordan and Wigner (see e.g. [1, 4]) for the case of a discrete quantum variable s.

Representation of the Second Type

Representations of the states of the field by functionals of the second type, described for boson fields in Section 19, can also be set up for fermion fields. We confine ourselves to describing representations by reduced functionals $\phi(\nu)$ with the aid of a measure function $w(\mathfrak{C})$. The space $\mathfrak{S}_w^F$ of fermion representers $\phi(\nu)$ can again be defined as the subspace of all functionals $\phi(\nu)$ in $\mathfrak{S}_w$ which are of the form

$$\phi(\nu) = F(\nu)\phi(\nu), \tag{21.17}$$

where $F(\nu)$ is the functional defined by (21.2); properly speaking, multiplication by $F(\nu)$ indicates application of the operator F obtained by extension of the operator which consists in multiplying by the proper functional $F(\nu)$. We recall that the operator F is a projector, satisfying the relation $F^2 = F$. The unit form (19.21) of the functionals $\phi(\nu)$ in $\mathfrak{S}_w$ may conveniently be written as

$$(\phi, \phi)_w = \sum_{\nu=0}' \mid \phi(\nu) \mid^2 \prod_s [dw(s)]^{\nu(s)} \exp\{-dw(s)\}, \tag{21.18}$$

again since $\nu(s)! = 1$ when $\nu(s) = 0$ or $\nu(s) = 1$.

In order to define annihilation and creation operators a weight amplitude $v^-(s)$ with $dw(s) = \mid v(s) \mid^2 dm(s)$ must be adopted as in Section 19. In accordance with (21.15) and (19.24) we then may set

$$A_w^-(s')\phi(\nu) = v^-(s')\tau(\nu, s')\phi(\nu + \delta_{s'}), \tag{21.19$^-$}$$

$$A_w^+(s'')\phi(\nu) = v^+(s'')[dw(s'')]^{-1}\tau(\nu - \delta_{s''}, s'')F(\nu)\phi(\nu - \delta_{s''}). \tag{21.19$^+$}$$

If the states Φ of a fermion field may be represented by functionals $\phi(\nu)$ in $\mathfrak{S}_w^F$ such that annihilation and creation operators $A_w^{\pm}(s')$ defined for them are represented by $A_w^{\pm}(s')$, we write

$$\Phi \overset{w}{\leftrightarrow} \phi(\nu). \tag{21.20}$$

Fermion fields whose states permit such a representation are called "myriotic" or "amyriotic" according as the total weight $w = \int dw(s)$ is infinite or finite. Amyriotic fields could be shown to be ordinary fields, i.e. fields which can be represented by functionals $f(\nu)$ of the first type; the connection is given by the same formula

$$f(\nu) = \phi(\nu) \prod_s [v^-(s)]^{\nu(s)} \exp\{-\tfrac{1}{2}w\} \tag{21.21}$$

as before, see (19.31).

It is not at all apriori obvious that myriotic fermion fields exist, inasmuch as it is not apriori obvious whether or not the space $\mathfrak{S}_w^F$ is the null space $\mathfrak{O}$, i.e. consists only of the functional $\phi(\nu) \equiv 0$. For it could happen that for every functional $\phi(\nu)$ in $\mathfrak{S}_w$ the functional $F(\nu)\phi(\nu)$ "equals" the functional zero, so

$F(\nu)$ "$=$" 0. We shall show below that this does indeed happen for certain weight functions $w(\mathfrak{C})$. At the same time we shall show that for other weight functions the space $\mathfrak{H}^F_w$ is not the null space. Thus we shall establish the existence of myriotic fermion fields.

Equidistribution State

With reference to the weight amplitude $v^-(s)$ we introduce the F-equidistribution state Φ^F_v by the representation

$$\Phi^F_v \overset{v}{\leftrightarrow} F(\nu). \tag{21.22}$$

Here $F(\nu) = F(\nu)1$ is the functional which results by applying the projector F on the representer $\phi(\nu) \equiv 1$ of the boson equidistribution state. Inserting $\phi(\nu) = F(\nu)$ into (21.18) we find in a formal manner

$$\begin{aligned} (F(\nu), F(\nu))_w &= \sum_{\nu=0}^{1} \prod_s [dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ (\Phi^F_v, \Phi^F_v) = (F(\nu), F(\nu))_w &= \prod_s [1 + dw(s)] \exp\{-dw(s)\}. \end{aligned} \tag{21.23}$$

The meaning of this expression will be explained below. The condition that it be positive is equivalent with the condition that $F(\nu) \not\approx 0$ or $\mathfrak{H}^F_w \not\approx \mathfrak{O}$.

The expectation value of the functional $\sum_s \zeta(s)N(s)$ in the F-equidistribution state is evidently the inner product of the functional $F(\nu)/\|\Phi^F_v\|$ with $\sum_s \zeta(s)\nu(s)F(\nu)/\|\Phi^F_v\|$. In a formal manner we evaluate

$$\begin{aligned} (F(\nu), \nu(s')F(\nu))_w &= \sum_{\nu=0}^{1} \nu(s') \prod_s [dw(s)]^{\nu(s)} \exp\{-dw(s)\} \\ &= dw(s') \exp\{-dw(s')\} \prod_{s \neq s'} [1 + dw(s)] \exp\{-dw(s)\} \\ &= [1 + dw(s')]^{-1} dw(s') \prod_s [1 + dw(s)] \exp\{-dw(s)\}. \end{aligned}$$

Combining this formula with (21.23) we find

$$\langle N(s')\rangle_{\Phi^F_v} = [1 + dw(s')]^{-1} dw(s') \tag{21.24}$$

whence

$$\langle \sum_s \zeta(s)N(s)\rangle_{\Phi^F_v} = \int \zeta(s)[1 + dw(s)]^{-1} dw(s). \tag{21.25}$$

The precise meaning of this peculiar expression will also be described below. The expected value of the total number $N = \sum_s N(s)$ of particles,

$$\langle N\rangle_{\Phi^F_v} = \int [1 + dw(s)]^{-1} dw(s) \tag{21.26}$$

will be shown to be finite or infinite according as $W = \int dw(s)$ is finite or infinite; i.e. according as the field is ordinary or myriotic.

Definition of the Functional $F(\nu)$

If the definition (21.4) of the functional $F(\nu)$ is taken literally it is not obvious what is meant by the functional $F(\nu)f(\nu)$ when the functional $f(\nu)$ is an ideal element in the space $\mathfrak{B}$. As indicated above, we interpret multiplication by $F(\nu)$ as application of a projector F defined in the following manner.

Let $\mathcal{P}$ be a partition and let $\mathcal{R}$ be the finite part of $\mathcal{P}$ composed of the bounded cells $\mathcal{C}_k$ of $\mathcal{P}$. Then we first define the functional $F_{\mathcal{P}}(\nu)$ by

$$(21.27) \qquad F_{\mathcal{P}}(\nu) \begin{cases} = 1 & \text{if} \quad \nu_k = 0 \quad \text{or} \quad = 1 \text{ for each } k \quad \text{and} \quad \nu_* = 0 \\ = 0 & \text{otherwise.} \end{cases}$$

Here, as before, $\nu_k = \sum_{s \text{ in } \mathcal{C}_k} \nu(s)$ and $\nu_* = \sum_{s \text{ in } \mathcal{C}^*} \nu(s)$. Let $f(\nu)$ be a functional which is compatible—in the first manner—with some partition $\mathcal{P}'$; by choosing a suitable refinement, we may assume that this partition is a subdivision of $\mathcal{P}$. Then the functional $F_{\mathcal{P}}(\nu)f(\nu)$ is also compatible with $\mathcal{P}'$. The operator which consists in multiplication by $F_{\mathcal{P}}(\nu)$ is thus defined for a manifold of functionals which is dense in the Hilbert space $\mathfrak{B}$. Since, evidently,

$$(21.28) \qquad \| F_{\mathcal{P}} f \| \leq \| f \|$$

this operator can be extended to the full Hilbert space. The operator $F_{\mathcal{P}}$ so defined will also be denoted as multiplication by $F_{\mathcal{P}}(\nu)$ even if it is applied to ideal elements $f(\nu)$. Clearly, this operator $F_{\mathcal{P}}$ is a projector, $F_{\mathcal{P}}^2 = F_{\mathcal{P}}$.

Let $\mathcal{P}'$ be a partition of $\mathcal{P}$ and $F_{\mathcal{P}'}$ the corresponding operator just defined. From (21.27) one verifies the relation

$$(21.29) \qquad F_{\mathcal{P}} F_{\mathcal{P}'} = F_{\mathcal{P}} ;$$

evidently this relation also holds for the operators applied to all of $\mathfrak{B}$. The operators $F_{\mathcal{P}'}$ corresponding to a succession of refinements of a partition thus form a non-decreasing sequence of projectors and, therefore, converge to a limit projector, see von Neumann [19, II, 4, Th. 17]. This limit projector F will also be denoted as "multiplication by $F(\nu)$."

Together with the projector F, the Hilbert space $\mathfrak{B}^F$ of functionals of the first type is now defined precisely.

Since the projector F is defined as the limit of a non-decreasing sequence it is clear that it is not identically zero,

$$(21.30) \qquad F \neq 0;$$

it is, however, not excluded that F is the identity. As a matter of fact, this is the case exactly when the quantum variable s has no point eigenvalues, as we shall show below.

If the operator F is to be applied to functionals $\phi(\nu)$ of the second type we must define it somewhat differently because of the different notion of compatibility involved. Let $\mathcal{R}$ be the finite part of a partition $\mathcal{P}$. Let $\mathcal{P}'$ be a refinement of $\mathcal{P}$ and let the subscript k' refer to those cells $\mathcal{C}_{k'}$ of $\mathcal{P}'$ which lie in $\mathcal{R}$. We first define the functional

$$(21.31) \qquad F'_{\mathcal{R}}(\nu) \begin{cases} = 1 & \text{if} \quad \nu_{k'} = 0 \quad \text{or} \quad = 1 \text{ for each } k' \\ = 0 & \text{otherwise.} \end{cases}$$

Let $\phi(\nu)$ be a function which is compatible in the second manner with some partition $\mathcal{P}''$ which without restriction may be assumed to be a refinement of $\mathcal{P}'$. Then the functional $F'_{\mathcal{R}}(\nu)\phi(\nu)$ is also compatible with $\mathcal{P}''$ in the second manner. Note that $F'_{\mathcal{R}}(\nu)$ does not depend on ν_* and hence not on $\nu''_* = \sum_{s \text{ in } \mathcal{C}''_*} \nu(s)$; the same is therefore true of $F'_{\mathcal{R}}(\nu)\phi(\nu)$.

As above it can be shown that a sequence of functionals $F'_{\mathcal{R}}(\nu)$ associated with a sequence of refinements of $\mathcal{P}$ forms a non-decreasing sequence of projectors and, therefore, converges to a limit which we denote by $F_{\mathcal{R}}$.

Now we consider a sequence $\mathcal{R}^\sigma$, $\sigma = 1, 2, \cdots$, of regions $\mathcal{R}$ which tend to cover the whole s-space. From (21.30) one readily concludes that relation

$$(21.32) \qquad F_{\mathcal{R}^\sigma} F_{\mathcal{R}^\tau} = F_{\mathcal{R}^\tau}.$$

holds if $\mathcal{R}^\sigma$ contains $\mathcal{R}^\tau$. Consequently, the operators $F_{\mathcal{R}^\sigma}$ form a non-increasing sequence of projectors and, therefore, converge to a limit projector F, which we also denote as multiplication by $F(\nu)$. It would not be difficult to show that this limit is independent of the choice of the sequence $\mathcal{R}^\sigma$.

Together with the projector F, also the Hilbert space of functionals of the second type $\mathfrak{H}^F_w$ is now defined precisely.

Since the operator F is defined as the limit of a non-increasing sequence of projectors it is not excluded that F is identically zero (we shall explain when this is the case). On the other hand, it is clear that $F \equiv 1$ only if $F_{\mathcal{R}} \equiv 1$ for all $\mathcal{R}$. This is the case exactly when the quantum variable s has no point eigenvalues.

We also note that for a myriotic field s the operators F defined in the first and second manner are identical.

Evaluations for the F-Equidistribution State

Since the operator F is now defined, also the functional

$$(21.33) \qquad F(\nu) = F1,$$

the representer of the F-equidistribution state Φ^F_*, is defined. We proceed to evaluate the norm $\| \Phi^F_* \| = \| F(\nu) \|_w$ of Φ^F_* and explain the meaning of the expression (21.23) by which this norm is expressed.

We assume that the point eigenvalues of the quantum variable s—if it has any—are so distributed that each bounded region contains at most a finite number of them. We denote these eigenvalues by $s^{(\rho)}$, $\rho = 1, 2, \cdots$, and by W_ρ the weight of the cell consisting of the point $s^{(\rho)}$.

We now let $\mathcal{P}'$ be a refinement of the partition $\mathcal{P}$ so chosen that each eigenvalue $s^{(\rho)}$ contained in $\mathcal{R}$ forms a cell of $\mathcal{P}'$. The notation $\mathcal{C}'_k$ will then be reserved for the remaining cells of $\mathcal{P}'$ contained in $\mathcal{R}$; the weight of these cells will be denoted by w'_k. From the definitions (19.21) and (21.31) we then have

$$(F'_{\mathcal{R}}(\nu), F'_{\mathcal{R}}(\nu))_w = \prod_\rho^{\mathcal{R}} (1 + W_\rho) \exp\{-W_\rho\} \prod_k{}' (1 + w'_k) \exp\{-w'_k\}.$$

Here the product $\prod^{\mathcal{R}}_\rho$ refers to those eigenvalues $s^{(\rho)}$ which lie in $\mathcal{R}$, and the product $\prod'_k$ refers to all cells $\mathcal{C}'_k$ in $\mathcal{R}$.

To every $\epsilon > 0$ we may choose a partition $\mathcal{P}'$ such that the weight w'_k of each cell $\mathcal{C}'_k$ in $\mathcal{R}$ is less than ϵ. From the elementary estimate

$$0 \le w - \log(1 + w) \le w^2 \le \epsilon w \qquad \text{for} \qquad 0 \le w \le \epsilon$$

we deduce

$$1 \ge \prod_k{}' (1 + w'_k) \exp\{-w'_k\} \ge \exp\{\epsilon \sum_k{}' w_k\}.$$

Now $\sum'_k w_k = w(\mathcal{R}) - \sum^{\mathcal{R}}_\rho W_\rho$ is the contribution to $w(\mathcal{R}) = \int_{\mathcal{R}} dw(s)$ from all cells in $\mathcal{R}$ with the exception of the point-eigenvalues; hence $\sum'_k w_k \le w(\mathcal{R}) < \infty$. Since ϵ is arbitrary we obtain for the limit $(F_{\mathcal{R}}(\nu), F_{\mathcal{R}}(\nu))_w$ of $(F'_{\mathcal{R}}(\nu), F'_{\mathcal{R}}(\nu))_w$ the evaluation

$$(21.34)_{\mathcal{R}} \qquad (F_{\mathcal{R}}(\nu), F_{\mathcal{R}}(\nu))_w = \prod_\rho^{\mathcal{R}} (1 + W_\rho) \exp\{-W_\rho\}.$$

We express this result also in the form

$$(21.35)_{\mathcal{R}} \qquad \prod_s^{\mathcal{R}} [1 + dw(s)] \exp\{-dw(s)\} = \prod_\rho^{\mathcal{R}} (1 + W_\rho) \exp\{-W_\rho\}.$$

Letting $\mathcal{R}$ expand so as to cover the whole s-space we find

$$(F(\nu), F(\nu))_w = \prod_\rho (1 + W_\rho) \exp\{-W_\rho\} \tag{21.36}$$

and

$$\prod_s [1 + dw(s)] \exp\{-dw(s)\} = \prod_\rho (1 + W_\rho) \exp\{-W_\rho\}, \tag{21.37}$$

with the provision that the right members stand for the number one in case the quantum variable s has no point eigenvalues. Thus the peculiar expression (21.25) is evaluated for $\zeta(s) \equiv 1$.

Each term in the product on the right of (21.37) is less than "one". If there are infinitely many point eigenvalues $s^{(\rho)}$ the product converges to a positive limit if and only if

$$\sum_\rho [W_\rho - \log(1 + W_\rho)] < \infty$$

or, what is equivalent, if

$$\sum_{\rho} W_{\rho}^{2} < \infty . \tag{21.38}$$

If, however

$$\sum_{\rho} W_{\rho}^{2} = \infty , \tag{21.38$_\infty$}$$

the infinite product has the value zero, i.e., we have

$$(F(\nu), F(\nu))_{w} = \prod_{s} [1 + dw(s)] \exp \{-dw(s)\} = 0. \tag{21.39}$$

The functional $F(\nu) = F1$ is zero in this case.

Identity (21.35) could be obtained by the following formal *rule*: *whenever a function $h(\alpha)$ vanishes of second order in α the sum of $h(dw(s))$ with respect to all s is*

$$\int h(dw(s)) = \sum_{\rho} h(W_{\rho}). \tag{21.40}$$

Formula (21.35) results for $h(\alpha) = \log (1 + \alpha) - \alpha$.

Other formulas could be obtained from this rule. In particular, one can obtain from it the evaluation of the "integral" $\int \zeta(s)[1 + dw(s)]^{-1} \, dw(s)$ entering formula (21.25). To this end we write

$$\int \zeta(s)[1 + dw(s)]^{-1} \, dw(s) = \int \zeta(s) \, dw(s) - \int \zeta(s)[1 + dw(s)][dw(s)]^{2}$$

and hence conclude

$$\int \zeta(s)[1 + dw(s)]^{-1} \, dw(s) = \int \zeta(s) \, dw(s) - \sum_{\rho} \zeta(s^{(\rho)})[1 + W_{\rho}]^{-1} W_{\rho}^{2} . \tag{21.41}$$

This expression may also be written in the form

$$\int \zeta(s)[1 + dw(s)]^{-1} \, dw(s) = \int' \zeta(s) \, dw(s) + \sum_{\rho} \zeta(s^{(\rho)})[1 + W_{\rho}]^{-1} W_{\rho} \tag{21.41$'$}$$

in which the prime attached to the integral sign indicates that the integration refers to all of the s-space except the point eigenvalues. Thus the peculiar expression (21.25) is evaluated for an arbitrary function $\zeta(s)$.

The expected value of the total number N of particles when the field is in the equidistribution state Φ_{w}^{F}, see (21.26), can now be expressed as

$$\langle N \rangle_{\Phi_{w}^{F}} = \int' dw(s) + \sum_{\rho} [1 + W_{\rho}]^{-1} W_{\rho} . \tag{21.42}$$

The right member here can be estimated by

$$\int [1 + dw(s)]^{-1} \, dw(s) \leq \int dw(s) = W, \tag{21.43}$$

and hence is finite if W is finite. In order to obtain a lower estimate we denote by r the number of eigenvalues $s^{(\rho)}$ with a weight $W_\rho > 1$. Indicating omission of these eigenvalues by a double prime we find

$$\int [1 + dw(s)]^{-1}\, dw(s) \geq \frac{1}{2} \int'' dw(s) + \frac{1}{2}\, r. \tag{21.43)'$$

Suppose now $W = \infty$. Then either $r = \infty$ or $\int'' dw(s) = \infty$; hence in any case $\int [1 + dw(s)]^{-1}\, dw(s) = \infty$. Thus the statement made in connection with (21.26) is reduced to the evaluation (21.41).

In order to justify this evaluation one should first assume the function to be piecewise constant, vanishing outside of a region $\mathfrak{R}$, and then approximate an arbitrary function by such piecewise constant functions. We shall not carry out details.

Properties of the Projector F

It was stated above that the projector F is the identity if the quantum variable s has no point eigenvalue. Since application of the projector F was symbolically written as multiplication by the functional $F(\nu)$ we may just as well say that this functional is identically equal to one if s has no point eigenvalue. This statement is, of course, more inclusive than the statement, implied by formula (21.36), that the functional $\phi(\nu) = F(\nu)1 = F(\nu)$ in $\mathfrak{B}^F$ is identically equal to the functional $\phi(\nu) \equiv 1$ if s has no point eigenvalue. The present contention; $F(\nu)\phi(\nu) = \phi(\nu)$ for every "boson" functional $\phi(\nu)$, i.e. for every functional in $\mathfrak{B}$ can, nevertheless, be deduced from the statement $F(\nu)1 = 1$ proved above.

Moreover, it is possible to deduce from this statement that the operator $F_{\mathfrak{R}}$, assigned to any region $\mathfrak{R}$, is the identity if the region $\mathfrak{R}$ contains no point eigenvalue of s.

In order to prove that relation

$$F_{\mathfrak{R}}\phi(\nu) = \phi(\nu) \tag{21.44}$$

holds for all functions $\phi(\nu)$ in $\mathfrak{B}$ if s has no point eigenvalue in $\mathfrak{R}$, it is sufficient to prove that this relation holds for all functionals of a dense subset in $\mathfrak{B}$. All compatible functionals form such a subset. In fact, a dense subset is already formed by all those compatible functionals

$$\phi(\nu) = \phi^{\mathcal{P}^\circ}(\overset{0}{\nu}_1, \overset{0}{\nu}_2, \cdots)$$

for which $\phi^{\mathcal{P}^\circ}$ is different from zero for only a finite number of systems $\overset{0}{\nu}_1, \overset{0}{\nu}_2, \cdots$. These functionals are linear combinations of those which are different from zero for only one such system $\overset{0}{\nu}_1 = \nu_1, \overset{0}{\nu}_2 = \nu_2, \cdots; \phi^{\mathcal{P}^\circ}(\nu^0) = \phi^0$.

Let $\phi_0(\nu)$ be such a functional. Without restriction we may assume that $\mathcal{P}^0$ is a refinement of the partition $\mathcal{P}$ of which $\mathfrak{R}$ is the finite part. Let $\mathcal{P}'$ be a refinement of $\mathcal{P}_0$ and let $F'_{\mathcal{R}}$ be the operator associated with it as defined by (21.31). We estimate the value of

$$|| (1 - F'_{\mathcal{R}})\phi_0 ||^2_w = \sum{}' \nu'_1 , \nu'_2 , \cdots | \phi_0(\nu) |^2 \prod_k \nu'_k ! [w'_k]^{\nu'_k} \exp \{-w'_k\}.$$

The prime attached to the summation sign indicates that one should omit all systems of occupation numbers ν'_1, ν'_2, $\cdots$ for which each ν'_k is either zero or one. Since the values of $\phi_0(\nu)$ that occur are either zero or ϕ^0 the value of the sum is increased if $\phi_0(\nu)$ is replaced by ϕ^0. Thus we find

$$|| (1 - F'_{\mathcal{R}})\phi_0 ||^2_w \leq | \phi^0 |^2 \sum_{\nu'_1, \nu'_2, \cdots}{}' \prod_k \nu'_k ! [w'_k]^{\nu'_k} \exp \{-w'_k\}$$

or,

$$|| (1 - F'_{\mathcal{R}})\phi_0 ||^2_w \leq | \phi^0 |^2 \, || (1 - F'_{\mathcal{R}})1 ||^2_w .$$

Since the refinement $\mathcal{P}'$ can be made arbitrarily fine this inequality leads to the inequality

$$|| (1 - F_{\mathcal{R}})\phi_0 ||^2_w \leq || (1 - F_{\mathcal{R}})1 ||^2_w .$$

Now,

$$|| (1 - F_{\mathcal{R}})1 ||^2_w = || 1 ||^2_w - || F_{\mathcal{R}}1 ||^2_w = 1 - || F_{\mathcal{R}}(\nu) ||^2 = 0,$$

as seen from $(21.34)_{\mathcal{R}}$, since it is assumed that there are no point eigenvalues in $\mathcal{R}$. Thus the content on $F_{\mathcal{R}}\phi(\nu) = \phi(\ )$ is proved.

The fact that the operator $F_{\mathcal{R}}$ is the identity when there is no point-spectrum of s in $\mathcal{R}$ indicates that we could have introduced the operator F in a different manner from the beginning. We could have defined $F_{\mathcal{C}}(\nu)$ by (21.4) if the cell $\mathcal{C}$ consists of just one point eigenvalue and by $F_{\mathcal{C}}(\nu) = 1$ if the cell $\mathcal{C}$ contains no eigenvalue of s; finally, we could have defined $F(\nu) = \sum_{\mathcal{C}} F_{\mathcal{C}}(\nu)$ when $\sum_{\mathcal{C}}$ refers to a decomposition of the whole s-space into bounded cells.

The properties of the functional $F(\nu)$ thus revealed might already have been used in connection with the occupation representation of boson fields. The values of $\nu(s)$ might have been restricted to $\nu(s) = 0$ and $\nu(s) = 1$ except for point eigenvalues. This fact was alluded to in Section 18 in the subsection on different forms of the occupation representations; it was also described there by saying that the oscillator assigned to a value of s is effectively excited at most to the first level. Nevertheless, it would seem that the general treatment of occupation representations which we have given is preferable.

Another consequence can be drawn from the fact that $F_{\mathcal{R}} = 1$ if $\mathcal{R}$ contains no point eigenvalue of s, namely that $F \not\equiv 0$ if the spectrum of s is not a pure point spectrum.

Suppose, on the other hand, the spectrum of s consists of only point eigenvalues $s^{(r)}$. Then the operator F vanishes identically if the field is myriotic,

$\sum_\rho w_\rho = \infty$, provided $\sum_\rho w_\rho^2 = \infty$; cf. $(21.38)_\infty$. In that case, therefore, a myriotic fermion field does not exist.

It follows from this statement, for example, that *no myriotic fermion field in a box exists in case every eigenstate of the energy carries the same weight.*

The statement

$$(21.45) \qquad F\phi(\nu) = 0$$

for all functions $\phi(\nu)$ *in* $\mathfrak{B}$ *under condition* $(21.38)_\infty$ *if* s *has a pure point spectrum* can be reduced to the statement, proved above, that the functional $F(\nu) = F1$ vanishes under this condition. We need only apply the argument employed to prove that $F = 1$ if s has no point spectrum. Let again $\phi_0(\nu)$ be a functional which equals ϕ^0 if $\nu_k = \nu_w^0$, and vanishes otherwise.

Then one finds first

$$\| F_{\mathfrak{R}}\phi_0 \|_w \leq | \phi^0 |^2 \| F_{\mathfrak{R}}1 \|_w ,$$

and then, by letting $\mathfrak{R}$ expand,

$$\| F\phi_0 \|_w^2 \leq | \phi^0 |^2 \| F_{\mathfrak{R}}1 \|_w = 0$$

by (21.39). Hence $F\phi(\nu) = 0$ for all $\phi(\nu)$ in $\mathfrak{S}_w$.

Partly Myriotic Fields

In conclusion we should like to call attention to certain types of states which are possible if the weight integral $\int dw(s)$ is infinite when extended over one part of the s-space and finite when extended over the complementary part of the s-space. Fields for which the weight function has this character will be called partly myriotic, although properly speaking they are myriotic since the total weight is infinite.

For simplicity we assume that one of the components of the quantum variable s may assume only a discrete number of variables; in fact, we assume that this component, denoted by ϵ, may assume only the values ±1. Furthermore, we change the notation by writing (s, ϵ) instead of s and assume that the measure $dm(s, \epsilon)$ is of such a form that

$$(21.46) \qquad \int \cdots dm(s, \epsilon) = \sum_{\epsilon=-1}^{1} \int \cdots dm_\epsilon(s).$$

Such a situation arises, for example, if the single particles are Dirac electrons when we take as quantum variable $s = (k, \tau)$ the momentum k and the component $\tau = (\sigma k)$ of the spin signature into the momentum, and identify ϵ with the sign of the energy $E = \epsilon \sqrt{\mu^2 + k^2}$.

We adopt a weight differential $dw(s, \epsilon)$ such that

$$(21.47) \qquad \int \cdots dw(s, \epsilon) = \sum_{\epsilon=-1}^{1} \int \cdots dw_\epsilon(s).$$

If then

$$\int dw_+(s) < \infty, \qquad \int dw_-(s) = \infty, \tag{21.48}$$

we may say that the field is myriotic as regards $\epsilon = -1$ and ordinary as regards $\epsilon = 1$.

For convenience, we shall set

$$\nu(s, \pm 1) = \nu_\pm(s), \qquad v(s, \pm 1) = v_\pm(s).$$

Since $\int dw_+(s)$ is finite, we may eliminate reference to the weight w_+ by introducing, instead of the functionals $\phi(\nu)$, the "hybrid" functionals

$$\digamma(\nu) = \prod_s [v_+^{-}(s)]^{\nu_+(s)} \phi(\nu), \tag{21.49}$$

cf. (19.15), so that the unit form becomes

$$\begin{aligned}(\digamma, \digamma)_{w-} = \sum_\nu \mid \digamma(\nu) \mid^2 \prod_s \nu_+(s)!\,[dm_+(s)]^{\nu_+(s)} \\ \cdot \prod_s \nu_-(s)!\,[dw_-(s)]^{\nu_-(s)} \exp\{-dw_-(s)\},\end{aligned} \tag{21.50}$$

cf. (19.3)′ and (19.22). The functionals $\digamma(\nu)$ are introduced so that no reference need be made to the weight $w_+(\mathfrak{C})$, neither in the expression for the unit form nor in the representation of the creation operator.

The particular state to which we want to call attention is the state Φ_{v-} which is represented by the functional

$$\digamma(\nu) \begin{cases} = F(\nu_-) & \text{if} \quad \nu_+(s) = 0 \quad \text{for all } s, \\ = 0 & \text{otherwise.} \end{cases} \tag{21.51}$$

Evidently, $(\digamma, \digamma)_{w+} = 1$ for this functional so that the state Φ_{v-} is normed. We realize that *the state* Φ_{v-} *represents a vacuum as regards* $\epsilon = 1$ *and an F-equidistribution state as regards* $\epsilon = -1$. We call Φ_{v-} a "semi-equidistribution state".

For the Dirac electrons referred to above we may, for instance, choose $dw_-(s) = dm_-(s)$ such that

$$\int \cdots dm_-(s) = \gamma \sum_{\tau=1}^{+1} \int \cdots dk \tag{21.52}$$

with an arbitrary positive number γ. Clearly, the weight of every region in momentum space is proportional to its volume, independently of the spin signature. Suppose now a field of Dirac electrons is in the semi-equidistribution state Φ_{1-}. Then with certainty no electron with positive energy will be found, while every region in momentum space is equally likely to be occupied by electrons with negative energy. The expected value of the number $N_{\mathfrak{R}-}$ of electrons with negative energy in a region $\mathfrak{R}$ of momentum space is

$$\langle N_{\mathfrak{R}-}\rangle_{\Phi_{1-}} = \gamma \mid \mathfrak{R} \mid \tag{21.53}$$

when $|\mathcal{R}|$ is the volume of $\mathcal{R}$. This fact follows from formula (19.41) for boson fields; for, we may set $F(\nu) = 1$ since the spectrum of s is continuous. The total number of electrons is hence expected to be infinite.

The probability that no particle with negative energy is present in the region $\mathcal{R}$ is different from zero. As seen from (19.41) it is

$$\Pr(N(\mathcal{R}_-) = 0) = \exp\{-\gamma\,|\,\mathcal{R}\,|\}. \tag{21.54}$$

Of course, this probability approaches zero as the region $\mathcal{R}$ expands so as to cover the whole field.

Suppose an observation of the number $N(\mathcal{R}_-)$ has been made and suppose the value $N(\mathcal{R}_-) = 0$ has been found. Then the field is in a new state in which with certainty the region $\mathcal{R}_-$ is not occupied. Thus the region $\mathcal{R}$ constitutes a "hole" in momentum space.

Naturally, one is tempted to consider the semi-equidistribution state Φ_{1-} as a *realization of Dirac's earlier idea of a uniform sea of electrons with negative energy* [2, Chapter XI, p. 73]. However, this is not completely justified since Dirac required that every region $\mathcal{R}_-$ is occupied with certainty, while in the semi-equidistribution state each region $\mathcal{R}_-$ may be unoccupied, with the probability $\exp\{-\gamma\,|\,\mathcal{R}\,|\}$. Nevertheless, it appears interesting that a notion of "sea of electrons with negative energy", similar to, but not identical with, that postulated by Dirac can be given a mathematically rigorous formulation.

BIBLIOGRAPHY TO PART IV

[37] Kolmogoroff, A., *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2, Heft 3, Julius Springer, Berlin, 1933.
[38] Tolman, R. C. *The Principles of Statistical Mechanics*, The Clarendon Press, Oxford, 1938.
[39] Fowler, R. H., *Staistical Mechanics*, Cambridge University Press, Cambridge, 1936.
[40] von Neumann, J., *Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren*, Mathematische Annalen, Volume 102, No. 1, 1929, pp. 49–131.
[41] Halmos P. R., *Introduction to Hilbert Space and the Theory of Spectral Multiplicity*, Chelsea Publishing Co., New York, 1951.

Part V. Fields Modified by Linear Homogeneous Forces

22. *Boson Fields under the Influence of Spring Forces*

In Part III we considered a boson field caused by the presence of a source distribution. We found that the source distribution contributes a right member to the differential equations for the field quantity $\Xi(x, t)$, making the equations non-homogeneous. In the present part we shall discuss the modifications of a field caused by forces which are linear in the field quantity Ξ; such forces will simply be called "spring forces." It is characteristic for this type of modification that the differential equation for the field quantity remains linear and homogeneous. The main result of this part is that the transformations needed to describe such modifications can be determined completely; in fact, they can be expressed explicitly in terms of the solution of the corresponding unquantized problem.

In the first sections of the present part we shall be concerned with boson fields, but in Section 28 we shall discuss similar problems for fermion fields, specifically, for fields of electrons and positrons endowed with Dirac's energy. Particular attention will be given to the problem of "vacuum polarization".

The boson field $\mathfrak{F}$ to be considered in the present section will be described by a field quantity or potential Ξ which is a function $\Xi(x, t)$ of the position x and the time t and satisfies the differential equation

$$(\nabla_t^2 - \nabla_x^2 + \mu^2)\Xi + Q^x\Xi = 0. \tag{22.1}$$

The symbol Q^x signifies a functional operator which acts on functions of x; it will be referred to as the "disturbance operator". The term $-Q^x\, \Xi(x, t)$ represents the force per unit mass exerted on the field at the point x at the time t.

The initial conditions for the solution $\Xi(x, t)$ of equation (22.1) are the same as those for the undisturbed field, cf. (2.1),

$$\begin{aligned} [\nabla_t\Xi(x', 0), \Xi(x'', 0)] &= -i\delta(x' - x''), \\ [\Xi(x', 0), \Xi(x'', 0)] &= 0, \\ [\nabla_t\Xi(x', 0), \nabla_t\Xi(x'', 0)] &= 0. \end{aligned} \tag{22.2}$$

The field $\mathfrak{F}$ may be an electromagnetic or a neutral meson field. However, it may also be an elastic medium provided the density and the elastic constants involved are taken equal to unity. In the following we usually refer to this interpretation. In an elastic medium, the quantity Ξ stands for the displacement or elongation. The position variable x may range either over the whole or a part of three-, two-, or one-dimensional space; if its range is only a part of one of these spaces, appropriate boundary conditions must be imposed on the functions of x on which the operator ∇_x^2 acts.

Adjoint and Conjugate Operators

Before formulating conditions on the disturbance operator Q we note that the notions of "real" adjointness and "real" symmetry will often be used instead of Hermitian adjointness and symmetry; also we shall assign to each operator a "complex conjugate". These notions are defined only with respect to a functional representation; we shall use the x-representation for this purpose.

The *complex conjugate* $\bar{\Lambda}$ of an operator Λ is defined by the condition that its x-representer $\bar{\Lambda}^x$ transforms a function $\psi(x)$ into the complex conjugate of the function $\Lambda^x \overline{\psi(x)}$,

$$(22.3) \qquad \bar{\Lambda}^x \psi(x) = \overline{\Lambda^x \overline{\psi(x)}}.$$

Here we assume that Λ^x is applicable to $\overline{\psi(x)}$ if it is applicable to $\psi(x)$. The *real adjoint* will be defined for operators Λ which have a Hermitian adjoint. Using the notation[1] $'\Lambda$ for the real adjoint and Λ^* for the Hermitian adjoint we define

$$(22.4) \qquad '\Lambda = \bar{\Lambda}^*.$$

Accordingly,

$$(22.5) \qquad \int {'\Lambda^x} \psi(x) \cdot \psi^{(1)}(x)\, dx = \int \psi(x) \Lambda^x \psi^{(1)}(x)\, dx$$

whenever Λ^x is applicable to $\psi(x)$, $\overline{\psi(x)}$, $\psi^{(1)}(x)$. If $'\Lambda = \Lambda$, we say that Λ is *real symmetric*. If $\bar{\Lambda} = \Lambda$ and $'\Lambda = \Lambda$, and hence $^*\Lambda = \Lambda$, we say that Λ is real *and* symmetric.

We now require that the disturbance operator Q be real and symmetric. More specific conditions on the nature of Q will be formulated later, see Section 26; but at present we may visualize Q^x as an integral operator, with a real and symmetric kernel $q(x, x')$,

$$(22.6) \qquad Q^x \psi(x) = \int q(x, x') \psi(x')\, dx'.$$

The force expressed by the term $-Q^x\psi(x)$ may be interpreted as the force due to a distribution of springs, connecting the points x' with the points x, the kernel $q(x, x')$ being the density of the spring constant.

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[1] Our notations differ from those customary in work on quantum physics.

Various Types of Problems

A number of generalizations of this problem may be treated in the same manner. These generalizations result from modification of the independent variable x, the differential dx, and the operator $\mu^2 - \nabla_x^2$. These modifications could in fact be subsumed under our theory by re-interpreting the terms x, dx, $\mu^2 - \nabla_x^2$ in an appropriate way.

One particularly interesting generalization is concerned with the interaction between several media; for example, between two two-dimensional membranes or two one-dimensional strings. In order to include such cases in our theory it is necessary to introduce instead of x a more general independent variable; we shall use the pair (x, ρ) in which ρ is a label running over the values $1, \cdots, r$ if r fields are considered. The integral $\int |\psi(x)|^2 dx$ is to be replaced by

$$\sum_{\rho=1}^{r} \int |\psi(x, \rho)|^2 dx.$$

Both the domain of integration and the rest mass μ may be different for each field. The term μ in (22.1) is accordingly to be replaced by μ^ρ and the operator Q^x by $Q^{x,\rho}$; this operator then incorporates the interaction between the various media.

Finally, cases in which masses are concentrated on two-, or one-dimensional manifolds or at single points may also be subsumed under the theory. In such cases the differential $dm(x)$ of an appropriate measure function must be substituted for dx and an appropriate undisturbed energy operator must be adopted in place of $\mu^2 - \nabla_x^2$.

For example, we may consider two "media", one extending over the whole space, the other concentrated at the point $x = 0$. The field quantity $\Xi(x, \rho)$ is then defined for all values of x when $\rho = 1$ but only for $x = 0$ if $\rho = 2$. Instead of $\Xi(x, \rho)$ we may use a pair $\{\Xi(x), \Xi_0\}$ to describe the field, letting $\Xi(x) = \Xi(x, 1)$ refer to $\rho = 1$ and $\Xi_0 = \Xi(0, 2)$ refer to $\rho = 2$. As the unit form for any pair $\{\psi(x), \psi_0\}$ we may choose

$$\int |\psi(x)|^2 dx + |\psi_0|^2.$$

The mass concentrated at the point $x = 0$ will be denoted by m but we assume that the rest mass of the extended field vanishes so that $(\Omega^x)^2 = -\nabla_x^2$ for this field. The corresponding term for the point mass is assumed to be of the form $\Omega_0^2 = c_0$ where c_0 is the constant of the spring which connects this mass with a support.

We suppose that the interaction of the extended field with the single mass is provided by springs with the density $q(x)$. Therefore, the interaction operator $Q^{x,\rho}$ transforms any pair $\{\psi(x), \psi_0\}$ into the pair

$$Q^{x,\rho}\{\psi(x), \psi_0\} = \left\{q(x)\psi_0, \int q(x)\psi(x)\, dx\right\}.$$

Evidently, this operator is real and symmetric.

Instead of (22.1) the differential equation becomes

$$(\nabla_t^2 - \nabla_x^2)\Xi + q(x)\Xi_0 = 0,$$

$$m\nabla_t^2\Xi_0 + \int q(x)\Xi(x)\,dx = 0;$$

and instead of (22.2) the initial conditions are

$$\nabla_t\Xi(x', 0)\cdot\Xi(x'', 0) - \Xi(x'', 0)\nabla_t\Xi(x', 0) = -i\delta(x' - x''),$$

$$m\nabla_t\Xi_0\cdot\Xi_0 - \Xi_0 m\nabla_t\Xi_0 = -i.$$

A special case of this problem is treated in the thesis of W. Sollfrey [50], namely the case in which the mass interacts with the medium only through a single spring attached to the point $x = 0$. Here

$$q(x) = c_1\delta(x),$$

if c_1 is the constant of the single spring. This point interaction makes the operator Q^x so singular that our treatment is not applicable. On the other hand, our treatment will explain the occurrence of infinities found by Sollfrey.

Modified and Unmodified Particle Representation

Instead of describing the field by means of the quantity $\Xi(x, t)$ we may describe it in terms of particles. With the aid of the energy operator

$$\Omega^x = [\mu^2 - \nabla_x^2]^{1/2} \tag{22.7}$$

we introduce the creation and annihilation operators

$$A^{\pm}(x, t) = (\Omega^x/2)^{1/2}\Xi(x, t) \mp i(2\Omega^x)^{1/2}\nabla_t\Xi(x, t), \tag{22.8}$$

the number operator

$$N(t) = \int A^+(x, t)A^-(x, t)\,dx, \tag{22.9}$$

and the energy operator

$$H_A(t) = \int A^+(x, t)\,\Omega^x A^-(x, t)\,dx, \tag{22.10}$$

see (9.5), (10.8), (10.9).

In order to justify the designation of $A^{\pm}(x, t)$ as creation and annihilation operators it is necessary to show that they satisfy the appropriate commutation laws and, furthermore, that a particle representation

$$\Phi \underset{N(t)}{\leftrightarrow} \{\psi_n(s)_n(t)\} \tag{22.11}$$

of the states of the field is associated with them.

The main object of our investigation is to determine the transition probability, i.e. the probability of finding n particles at the time t, or as $t \to \infty$, if no particles were present at the time $t = 0$.

Neither the number operator N, nor the energy operator H_A is constant in time. However, the sum of the energy operator $H_A(t)$ and the interaction operator $V(t)$, the total energy operator

$$H_{\text{tot}} = H_A(t) + V(t), \tag{22.12}$$

is constant. The interaction operator $V(t)$ is given by

$$\begin{aligned} V(t) &= \frac{1}{2} \int \Xi(x, t) Q^x \Xi(x, t)\, dx \\ &= \frac{1}{4} \int [A^+(x, t) + A^-(x, t)](\Omega^x)^{-1/2} Q^x (\Omega^x)^{-1/2} [A^+(x, t) + A^-(x, t)]\, dx. \end{aligned} \tag{22.13}$$

The constancy of the operator H_{tot} is easily verified formally from the differential equations (21.1).

Whether or not the operator $V(t)$ is defined depends on the nature of the operator Q. It could be shown that $V(t)$ can be defined if $\Omega^{-1/2}\, Q\, \Omega^{-1/2}$ is an integral operator with a quadratically integrable kernel. Since we shall not actually work with the operator $V(t)$ no details will be given.

Modified Energy Operator for Single Particles

It is our aim to introduce a "modified" particle representation of the states of the field such that H_{tot} is the associated energy operator except for an additive constant.

The modified particles are endowed with an energy operator which differs from the energy operator Ω^x associated with the unmodified particles. The simplest way to determine the proper expression of the modified particle energy operator is to compare the unmodified and modified field energy operators in the improper forms

$$\frac{1}{2} \int [\nabla_t \Xi^2 + \Xi(\mu^2 - \nabla_x^2)\Xi]\, dx$$

and

$$\frac{1}{2} \int [(\nabla_t \Xi)^2 + \Xi(\mu^2 - \nabla_x^2)\Xi + \Xi Q^x \Xi]\, dx.$$

Clearly, the second expression is of the same form as the first, but we have

$$\Upsilon^x = (\mu^2 - \nabla_x^2 + Q^x)^{1/2} \tag{22.15}$$

in place of $\Omega^x = (\mu^2 - \nabla_x^2)^{1/2}$, see (22.7). This operator Υ is defined if the operator $\mu^2 - \nabla_x^2 + Q^x$ is non-negative; if it is positive definite, the inverse Υ^{-1} of Υ is defined. We assume that the disturbance Q is such that this is the case.

We suppose that the spectral representation of the operator Υ is "known". We may then consider all functions of Ω and Υ as known. At the beginning of this part we stated that our problem can be solved explicitly; this statement can now be made specific: we shall solve our problem explicitly in terms of the operators Ω and Υ.

Employing the operator Υ we can interpret our problem with reference to the single particles of which we may consider the field to be composed. Instead of saying that forces are exerted on the field we may say that forces act on the single particles. Instead of describing this force explicitly it is sufficient to describe the potential energy associated with it. The operator which corresponds to this potential energy is simply the difference $\Upsilon - \Omega$. We may also say that the disturbance is such that Υ has replaced Ω as energy operator.

Modified Creation and Annihilation Operators

In terms of the modified energy operator Υ we may introduce "modified" creation and annihilation operators by

$$B^{\pm}(x, t) = (\Upsilon^x/2)^{1/2}\Xi(x, t) \mp i(2\Upsilon^x)^{-1/2}\nabla_t\Xi(x, t); \tag{22.15}$$

the associated "modified" number operator is

$$M = \int B^{+}(x, t)B^{-}(x, t)\, dx \tag{22.16}$$

and the "modified" energy operator is

$$H_B = \int B^{+}(x, t)\Upsilon^x B^{-}(x, t)\, dx. \tag{22.17}$$

The differential equation (22.1) for the operators Ξ leads to the equation

$$\nabla_t B^{\pm}(x, t) = \pm i\Upsilon^x B^{\pm}(x, t) \tag{22.18}$$

for the operators $B^{\pm}$. Evidently the solution is

$$B^{\pm}(x, t) = \exp\{\pm it\Upsilon^x\}B^{\pm}(x, 0), \tag{22.19}$$

provided that the initial values $B^{\pm}(x, 0)$ are given.

Of course, it is assumed that these initial values obey the commutation laws; it then follows that the $B^{\pm}(x, t)$ also obey these laws.

The constancy of M and H_B is an immediate consequence of (22.19).

We may now express the solution $\Xi(x, t)$ of equation (22.1) in terms of the initial data $\Xi(x, 0)$, $\nabla_t\Xi(x, 0)$ by inverting relation (22.15) on the assumption that this relation holds initially. It can then be verified that $\Xi(x, t)$ and $\nabla_t\Xi(x, t)$ obey the commutation laws for each value of t.

With the aid of (22.8) we can finally express the operators $A^{\pm}(x, t)$ in terms of their initial values. One then easily verifies that these operators obey the commutation laws.

It is necessary to express the operators $A^{\pm}$ directly in terms of the operators $B^{\pm}$. Eliminating Ξ and $\nabla_t \Xi$ from relations (22.15) and (22.9) we find first

$$\begin{aligned} B^{+}(x, t) + B^{-}(x, t) &= (\Upsilon^{1/2}\Omega^{-1/2})^{x}[A^{+}(x, t) + A^{-}(x, t)], \\ B^{+}(x, t) - B^{-}(x, t) &= (\Upsilon^{-1/2}\Omega^{1/2})^{x}[A^{+}(x, t) - A^{-}(x, t)], \end{aligned} \tag{22.20}$$

and consequently,

$$B^{\pm}(x, t) = Y_{+}^{x}A^{\pm}(x, t) + Y_{-}^{x}A^{\mp}(x, t), \tag{22.21}$$

if we introduce the operators $Y_{\pm}$ by

$$2Y_{\pm} = \Upsilon^{1/2}\Omega^{-1/2} \pm \Upsilon^{-1/2}\Omega^{1/2}. \tag{22.22}$$

Introducing the operators $Z_{\pm}$ by

$$2Z_{\pm} = \Omega^{1/2}\Upsilon^{-1/2} \pm \Omega^{-1/2}\Upsilon^{1/2}, \tag{22.23}$$

we have inversely

$$A^{\pm}(x, t) = Z_{+}^{x}B^{\pm}(x, t) + Z_{-}^{x}B^{\mp}(x, t). \tag{22.24}$$

Note that the modified and unmodified creation and annihilation operators are related by linear homogeneous relations; this will be the starting point of our procedure in determining the modified particle representation.

Remarks About Energy Operators

At present we add a few remarks concerning the relationship between the energy operators H_B and H_{tot}, see (22.17) and (22.12). Inserting the expressions (22.24) into (22.12) and using the commutation laws we find

$$H_{\text{tot}} = H_B + h. \tag{22.25}$$

The term h is an integral which can be written in the form

$$h = \frac{1}{4}\int [A^{-}(x)J^{x}A^{+}(x) - A^{+}(x)J^{x}A^{-}(x)]\, dx$$

when the operator J is defined by

$$J = J_{+} + J_{-},$$

$$J_{\pm} = \Omega^{\pm 1/2}[\Upsilon - \Omega]\Omega^{\mp 1/2}.$$

Suppose now the operators $J_{\pm}^{x}$ are integral operators with kernels $j_{\pm}(x', x'')$ which possess finite traces

$$Tr\, J_{\pm} = \int j_{\pm}(x, x)\, dx.$$

Then we find

$$h = \tfrac{1}{4}\, Tr\, J_{+} + \tfrac{1}{4}\, Tr\, J_{-}.$$

We may give this expression a different form by using the fact that $Tr\, AB = Tr\, BA$ for any two operators. Supposing that the operator $\Upsilon - \Omega$ possesses a trace we have

$$\begin{aligned} Tr\, J_{\pm} &= Tr\, \Omega^{\pm 1/2}(\Omega^{\mp 1/2} J) \\ &= Tr\, (\Omega^{\mp 1/2} J)\Omega^{\pm 1/2} = Tr\, (\Upsilon - \Omega); \end{aligned}$$

hence

$$h = \tfrac{1}{2}\, Tr\, (\Upsilon - \Omega). \tag{22.26}$$

The condition that this trace must be finite involves a severe restriction on the nature of the operator $Q = \Upsilon^2 - \Omega^2$. Closely related conditions on the disturbance Q will be formulated in Section 26.

Modified Particle Representation

Our main objective is to determine the particle representation (22.13)

$$\Phi \underset{N(t)}{\leftrightarrow} \{\psi_n(x)_n(t)\} \tag{22.27}$$

associated with the operators $A^{\pm}(x, t)$ if the representation of the state is given for the initial time $t = 0$. To this end we use the modified particle representation

$$\Phi \underset{M(t)}{\leftrightarrow} \{\chi_m(x_m)(t)\}.$$

In analogy with the procedure described in Part III, cf. (14.10), we determine a transformation T_M^N and its inverse T_N^M which transform the functions $\psi_m(x)_m(t)$ and $\chi_m(x)_m(t)$ into each other,

$$\begin{aligned} \chi_m(s)_m(t) &= T_M^N \psi_n(s)_n(t), \\ \psi_n(s)_n(t) &= T_N^M \chi_m(s)_m(t). \end{aligned} \tag{22.28}$$

Suppose the transformations T_N^M, T_M^N are known. Then we may obtain $\chi_m(x)_m(0)$ from

$$\chi_m(x)_m(0) = T_M^N \psi_n(x)_n(0);$$

furthermore,

$$\chi_m(x)_m(t) = \exp\{-itH_B\}\chi_m(x)_m(0)$$

from (22.19), and finally

$$\psi_n(x)_n(t) = T_N^M \chi_m(x)_m(t).$$

Combining these formulas we find

$$\psi_n(x)_n(t) = T_N^M \exp\{-itH_B\} T_M^N \psi_n(x)_n(0). \tag{22.29}$$

The transition probabilities can then be calculated.

The transformation (22.8) will also be determined directly, without reference to modified particles as intermediaries.

Canonical Transformations

As in Part III we shall derive the transformation T with the aid of a canonical transformation, see Section 14. We introduce the "trivial" transformation I_N^M which transforms the M-representers $\chi_m(x)_m$ of any state Φ into the N-representers

$$\chi_n(x)_n = I_N^M \chi_m(x)_m$$

of that state Φ' whose N-representers happen to be the same functions as the M-representers of Φ, cf. (14.17). We wish to find the unitary transformation T which transforms the state Φ into the states Φ',

$$T\Phi = \Phi'. \tag{22.30}$$

Let T^N be the N-representer of the operator T; then the M-representers of the state $\Phi = T^{-1}\,\Phi'$ are given by

$$\chi_m(x)_m = I_M^N T^N \psi_n(x)_n \tag{22.31}$$

in terms of the N-representers of Φ. In other words, the desired transformation of the N-representers into the M-representers of a state Φ is given by

$$T_M^N = I_M^N T^N, \tag{22.32}$$

cf. (14.22). Our problem has thus been reduced to the determination of the unitary operator T.

The transformation of the modified vacuum state Φ_{vac}^B into the modified vacuum state Φ_{vac}^A is of particular interest. Since the N-representers of Φ_{vac}^A are the same as the M-representers of Φ_{vac}^B, namely, $\{1, 0, 0, \cdots\}$, it is clear from (22.30) that these states are related by

$$\Phi_{\text{vac}}^A = T\Phi_{\text{vac}}^B\,. \tag{22.33}$$

In fact, the knowledge of the transformation of Φ_{vac}^B into Φ_{vac}^A expressed in terms of the creation operators B^+ would already enable one to construct the general transformation of M- into N-representers. It is, however, much simpler to give the complete transformation T directly.

As in Section 14, we shall characterize the operator T, without reference to representers, by the condition

$$A^{\pm}(x')T = TB^{\pm}(x') \tag{22.34}$$

cf. (14.21). We shall determine the unitary operator T so that it satisfies this condition and define the representers $\chi_m(x)_m$ of a state Φ by (22.31). It is clear that the operators $B^{\pm}(x)$ are exactly the creation and annihilation operators associated with the representation of Φ in terms of these functions $\chi_m(x)_m$.

23. *General Homogenous Linear Transformation of Creation and Annihilation Operators*

Our procedure for determining a unitary operator T which satisfies relation (22.35) will be developed quite generally under the assumption that the operators $B^{\pm}$ and $A^{\pm}$ are connected by any homogeneous linear transformation which guarantees that the operators B^{+} and B^{-} are conjugate to each other and obey the commutation laws provided the operators A^{+} and A^{-} have these properties. Relation (22.21) is just a special case of such a transformation. In fact, we shall develop our procedure simultaneously for fermion and boson annihilation and creation operators.

We shall also adopt a general quantum variable s with a measure differential $dm(s)$ instead of x and dx.

It is convenient to consider the pair of operators $A^{\pm}(s)$ as one entity

$$\mathfrak{A}(s) = \{A^{-}(s), A^{+}(s)\}; \tag{23.1}$$

more generally, to a pair of functions $v^{\pm}(s)$ in $\mathfrak{H}$, i.e., to a pair of quadratically integrable functions, we assign the "pseudo-function"

$$v(s) = \{v^{-}(s), v^{+}(s)\}. \tag{23.2}$$

To two such pseudo-functions, $v(s)$ and $w(s)$, we assign the "pseudo-product"

$$w \circ v = \int [w^{+}(s)v^{-}(s) \mp w^{-}(s)v^{+}(s)]\, dm(s). \tag{23.3}$$

Henceforth the upper sign is to be used if the operators $A^{\pm}$ refer to a boson field, the lower sign if the $A^{\pm}$ refer to a fermion field. Evidently,

$$v \circ w = \mp(w \circ v) \tag{23.3$'$}$$

We also use commutators and anticommutators

$$[\Lambda_1 ; \Lambda_2] = \Lambda_1\Lambda_2 \mp \Lambda_2\Lambda_1 \tag{23.4}$$

in connection with boson and fermion fields respectively (the semicolon is used in order to make it unnecessary to indicate the anticommutator by a special subscript).

With the aid of functions $v^{\pm}$ and $w^{\pm}$ belonging to appropriate subspaces of $\mathfrak{H}$ we may form the field operators[2] $w \circ \mathfrak{A}$ and $\mathfrak{A} \circ v$; c.f. (8.5), (8.13). One readily verifies that the commutation laws (8.16), (8.20) and (8.16)$^{\pm}$, (8.20)$^{\pm}$ are equivalent with the validity of the relation

$$[w \circ \mathfrak{A}; \mathfrak{A} \circ v] = w \circ v \tag{23.5}$$

or

$$[\mathfrak{A} \circ w; v \circ \mathfrak{A}] = w \circ v \tag{23.5$'$}$$

for arbitrary pseudo-functions v and w.

---

[2] A field operator acts on states of the field. We shall use this term whenever it is desirable to distinguish such operators from the "operators" which act on states of single particles.

With the aid of four bounded operators $L^S_{\pm\pm}$ acting on functions in $\mathfrak{H}$ we form the "pseudo-operator"

$$\mathcal{L}^S = \begin{pmatrix} L^S_{--} & L^S_{-+} \\ L^S_{+-} & L^S_{++} \end{pmatrix} \tag{23.6}$$

which transforms the pseudo-function $v(s)$ into the pseudo-function

$$\mathcal{L}^S v(s) = \{L^S_{--}v^-(s) + L^S_{-+}v^+(s),\ L^S_{+-}v^-(s) + L^S_{++}v^+(s)\}. \tag{23.7}$$

We write the pseudo-product of $w(s)$ and $\mathcal{L}^S\, v(s)$ as $w \circ \mathcal{L}\, v$, omitting the superscript S; in fact, we shall omit this superscript whenever possible without inconsistency. To a pseudo-operator $\mathcal{L}$ we assign its "pseudo-adjoint" ${}'\mathcal{L}$ by the condition

$$\mathcal{L}w \circ v = w \circ {}'\mathcal{L}v. \tag{23.8}$$

Evidently,

$${}'\mathcal{L}^S = \begin{pmatrix} {}'L^S_{++} & \mp{}'L^S_{-+} \\ \mp{}'L^S_{+-} & {}'L^S_{--} \end{pmatrix}. \tag{23.9}$$

Here the operators ${}'L^S_{\pm\pm}$ are the real adjoints of the operators $L^S_{\pm\pm}$. To appropriate functions $f(\lambda)$ we may assign the pseudo-operators $f(\mathcal{L})$; clearly

$${}'f(\mathcal{L}) = f({}'\mathcal{L}). \tag{23.10}$$

To the pair of pseudo-operators $\mathcal{L}$, $\mathcal{M}$ we may assign the commutator $[\mathcal{L}, \mathcal{M}]$. Note that the symbol [,] indicates the proper commutator

$$[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}. \tag{23.11}$$

Evidently,

$${}'[\mathcal{L}, \mathcal{M}] = -[{}'\mathcal{L}, {}'\mathcal{M}]. \tag{23.12}$$

A pseudo-operator $\mathcal{P}$ will be called "pseudo-antisymmetric" if

$${}'\mathcal{P} = -\mathcal{P}, \tag{23.13}$$

i.e. if

$${}'P_{++} = -P_{--}\,, \qquad {}'P_{--} = -P_{++}\,, \tag{23.14$_1$}$$

$${}'P_{+-} = \pm P_{+-}\,, \qquad {}'P_{-+} = \pm P_{-+}\,. \tag{23.14$_2$}$$

One important fact about pseudo-antisymmetric operators is that *the commutator* $[\mathcal{P}_1, \mathcal{P}_2]$ *of two such operators is also pseudo-antisymmetric.* From relation (23.10) it can be inferred, moreover, that the matrix exp $\mathcal{P}$ has the property

$${}'(\exp \mathcal{P}) = (\exp \mathcal{P})^{-1} \tag{23.15}$$

if $\mathcal{P}$ is pseudo-antisymmetric.

We shall frequently impose an additional condition on the pseudo-operators $\mathcal{L}$, namely

$$\overline{L}_{\sigma,\tau} = L_{-\sigma,-\tau} . \tag{23.16}$$

Here σ and τ stand for the labels + or −, and $\overline{L}$ is the complex conjugate of L. Pseudo-operators $\mathcal{L}$ whose components satisfy condition (23.16) will be called "pseudo-hermitian." If a pseudo-operator $\mathcal{L}$ is speudo-hermitian every real function of $\mathcal{L}$ is also pseudo-hermitian.

We now assume that a linear transformation of the "unmodified" annihilation and creation operators $\mathcal{A} = \{A^-(x), A^+(x)\}$ into modified operators $\mathcal{B} = \{B^-(x), B^+(x)\}$ of the form

$$\mathcal{B} = \mathcal{Y}\mathcal{A} \tag{23.17}$$

is given, such that the pseudo-operator $\mathcal{Y}$ is pseudo-hermitian and satisfies the condition

$$\mathcal{Y}'\mathcal{Y} = {}'\mathcal{Y}\mathcal{Y} = 1 \tag{23.18}$$

or

$${}'\mathcal{Y} = \mathcal{Y}^{-1}. \tag{23.18$'$}$$

By assumption, the hermitian adjoint of the operator A^σ is $(A^\sigma)^* = A^{-\sigma}$. Hence the hermitian adjoint of the operator $B^\sigma = Y_{\sigma,-}A^- + Y_{\sigma,+}A^+$ is $(B^\sigma)^* = \overline{Y}_{\sigma,-}A^+ + \overline{Y}_{\sigma,+}A^- = Y_{-\sigma,-}A^- + Y_{-\sigma,+}A^+ = B^{-\sigma}$. Thus the pseudo-hermitian character of $\mathcal{Y}$ insures that *the operators $B^\pm(x)$ are hermitian adjoints of each other.*

We also maintain that condition (23.18), imposed on the pseudo-operator $\mathcal{Y}$, insures that *the operators $B^\pm(x)$ satisfy the commutation laws if*, as assumed, the operators $A^\pm(x)$ do. In fact, from (23.8), (23.5), and (23.18) we deduce

$$[w \circ \mathcal{B}; \mathcal{B} \circ v] = [{}'\mathcal{Y}w \circ \mathcal{A}; \mathcal{A} \circ {}'\mathcal{Y}v] = {}'\mathcal{Y}w \circ {}'\mathcal{Y}v = w \circ \mathcal{Y}'\mathcal{Y}v$$

and therefore relation $[w \circ \mathcal{B}; \mathcal{B} \circ v] = w \circ v$; which is equivalent to the commutation laws.

Suppose a pseudo-antisymmetric pseudo-hermitian operator $\mathcal{R}$ exists such that

$$\mathcal{Y} = \exp \mathcal{R}. \tag{23.19}$$

Then $\mathcal{Y}$ is also pseudo-hermitian and we conclude that in relation (23.15) the operator $\mathcal{Y} = \exp \mathcal{R}$ satisfies the condition $\mathcal{Y}'\mathcal{Y} = 1$, cf. (23.18). In the following we assume that $\mathcal{Y}$ is of this form.

The canonical transformation T will first be described in terms of the pseudo-operator $\mathcal{R}$. Eventually, however, we shall describe this transformation in different ways independently of the assumption that the pseudo-operator $\mathcal{R}$ exists. Incidentally, for fermion fields the existence of the pseudo-operator $\mathcal{R}$ is always certain since in this case the pseudo-operator $\mathcal{Y}$ is unitary.

The transformation (23.15) of operators $A^\pm$ into operators $B^\pm$, connected with the modification of a boson field under the influence of spring forces, is of the form (23.17) with a pseudo-operator

$$\mathcal{Y} = \begin{pmatrix} Y_+ & Y_- \\ Y_- & Y_+ \end{pmatrix} \tag{23.20}$$

whose components are given by

$$\begin{aligned} Y_{--} &= Y_{++} = Y_+ = \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} + \Upsilon^{-1/2}\Omega^{1/2}], \\ Y_{+-} &= Y_{-+} = Y_- = \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} - \Upsilon^{-1/2}\Omega^{1/2}], \end{aligned} \tag{23.21}$$

according to (23.16). Clearly, this pseudo-operator $\mathcal{Y}$ satisfies condition (23.16) since Υ and Ω are real. Also, the pseudo-adjoint $'\mathcal{Y}$ of $\mathcal{Y}$ is given by

$$'\mathcal{Y} = \begin{pmatrix} Z_+ & Z_- \\ Z_- & Z_+ \end{pmatrix} \tag*{(23.20)$'$}$$

with

$$\begin{aligned} Z_+ &= \tfrac{1}{2}[\Omega^{-1/2}\Upsilon^{1/2} + \Omega^{1/2}\Upsilon^{-1/2}] \\ Z_- &= \tfrac{1}{2}[\Omega^{-1/2}\Upsilon^{1/2} - \Omega^{1/2}\Upsilon^{-1/2}]. \end{aligned} \tag*{(23.21)$'$}$$

As seen from (22.23), the operator $'\mathcal{Y}$ is the inverse of $\mathcal{Y}$, so that condition (23.18)′ is satisfied. It is not obvious, though, whether or not this pseudo-operator $\mathcal{Y}$ is of the form $\mathcal{Y} = \exp \mathcal{R}$. In constructing the canonical transformation T we shall at first assume that it is, but this assumption will be eventually discarded; for, we shall be able to verify the correctness of the final expression of T independently.

Pseudo-biquantized Operators

Our procedure for finding a canonical transformation T explicitly makes extensive use of certain field operators $[\mathcal{P}]$ which are assigned to pseudo-antisymmetric pseudo-operators $\mathcal{P}$ by the formula

$$\begin{aligned} [\mathcal{P}] &= \tfrac{1}{2}\mathcal{A} \circ \mathcal{P}\mathcal{A} \\ &= \tfrac{1}{2}A^+P_{--}A^- + \tfrac{1}{2}A^+P_{-+}A^+ \mp \tfrac{1}{2}A^-P_{+-}A^- \mp \tfrac{1}{2}A^-P_{++}A^+ \\ &= \tfrac{1}{2}\int A^+(s)P^s_{--}A^-(s)\,dm(s) + \tfrac{1}{2}\int A^+(s)P^s_{-+}A^+(s)\,dm(s) \\ &\qquad \mp \tfrac{1}{2}\int A^-(s)P^s_{+-}A^-(s)\,dm(s) \mp \tfrac{1}{2}\int A^-(s)P^s_{++}A^+(s)\,dm(s). \end{aligned} \tag{23.22}$$

This assignment, which is a basic tool in the following investigations, will be studied in detail. It is analogous to the assignment of the biquantized operator

A^+PA^- to the single particle operator P. We shall, therefore, call the operator $[\mathcal{P}]$ "pseudo-biquantized". For this assignment we are using the same bracket notation we used for the proper biquantized operator cf. (6.35); no confusion is to be expected.

The pseudo-biquantized operator $[\mathcal{P}]$ is not defined for arbitrary pseudo-antisymmetric pseudo-operators $\mathcal{P}$; the terms $P_{\cdot\cdot}$ of this pseudo-operator must satisfy appropriate conditions. The term $A^+P_{--}A^-$, the first contribution to $2[\mathcal{P}]$, is the biquantization of P_{--} ; hence it is defined whenever P_{--} is. The last term $\mp A^-P_{++}A^+$, however, differs from the biquantization of P_{++} by the trace of $\mp P_{++}$,

$$\mp A^-P_{++}A^+ = A^+{}'P_{++}A^- \mp Tr\,P_{++}\ , \tag{23.23}$$

cf. Section 10. If P^S_{++} is an integral operator with the kernel $P_{++}(s', s'')$, the trace TrP_{++} , provided it exists, can be expressed as

$$Tr\,P_{++} = \int P_{++}(s, s)\ dm(s). \tag{23.24}$$

The trace does exist if

$$\int |\ P_{++}(s, s)\ |\ dm(s) < \infty; \tag{23.25}$$

this condition evidently implies a severe restriction on the operator P_{++} . From the expression (23.24) we infer that the trace of the operator P_{++} equals that of its transposed $'P_{++}$,

$$Tr\,P_{++} = Tr\,'P_{++}\,. \tag{23.26}$$

Since $'P_{++} = -P_{--}$ in virtue of the pseudo-antisymmetry of $\mathcal{P}$, cf. (23.14) the contribution (23.23) can also be written as

$$\mp A^-P_{++}A^+ = A^+P_{--}A^- \pm Tr\,P_{--}\,. \tag{23.27}$$

We take the liberty of calling an operator Λ "traceable" if its trace exists; we shall call it "square traceable" if the trace of the operator $\Lambda^*\Lambda$ exists. If Λ is square traceable its s-representer Λ^S is an integral operator with a kernel $\Lambda(s', s'')$ for which

$$Tr\,\Lambda^*\Lambda = \iint |\ \Lambda(s', s'')\ |^2\ dm(s')\ dm(s''). \tag{23.28}$$

Evidently,

$$Tr\,\Lambda\Lambda^* = Tr\,\Lambda^*\Lambda. \tag{23.29}$$

The contribution $A^-P_{+-}A^-$ to the field operator $2[\mathcal{P}]$ is defined if P^S_{+-} is an integral operator with a kernel $P_{+-}(s', s'')$ for which

$$\iint |\ P_{+-}(s', s'')\ |^2\ dm(s')\ dm(s'') < \infty. \tag*{(23.30)$_1$}$$

This condition can now be expressed by saying that the operator P_{+-} should be square traceable. Clearly, the operator $A^-P_{+-}A^-$ transforms the two particle state Φ_2 with the representer $\psi_2(s_1, s_2)$ into the vacuum state with the representer $\psi_0 = \sqrt{2} \int\int P_{+-}(s', s'') \psi_2(s', s'') \, dm(s') \, dm(s'')$, and this term is finite if condition $(23.30)_1$ is satisfied. One immediately verifies that under this condition the operator $A^-P_{+-}A^-$ is applicable to every n-particle state. To be sure, condition (23.26) is more severe than necessary. On the other hand, the operators P_{+-} (which will occur in the following) will be closely connected with the operators P_{-+}, and the condition

$$(23.30)_2 \qquad \iint |P_{-+}(s', s'')|^2 \, dm(s') \, dm(s'') < \infty$$

must be imposed on the kernel $P_{-+}(s', s'')$ of the operators P_{-+}; i.e. the operators P_{-+} must be square traceable. In fact, the operator $A^+P_{-+}A^+$ transforms the vacuum state with the representer $\psi_0 = 1$ into the two particle state with the representer

$$\psi_2(x_1, x_2) = \sqrt{2}\, P_{-+}(s_1, s_2)$$

and this state exists only if condition (23.29) is satisfied. However, it is easily verified that this condition is sufficient to insure that the operator $A^+P_{-+}A^+$ be applicable on every n-particle state.

We note that actually only the symmetric parts of P_{+-} and P_{-+} contribute to the operator $[\mathcal{P}]$ in case of a boson field, and only the antisymmetric parts in case of a fermion field; but for pseudo-antisymmetric pseudo-operators $\mathcal{P}$ the operators P_{+-} and P_{-+} are symmetric—or antisymmetric—see $(23.14)_2$.

Under the conditions imposed the field operator $[\mathcal{P}]$ can be applied to at least all states Φ which admit the number operator $N = A^+A^-$, see (10.8). For, as would not be difficult to show, a constant γ can be found such that the inequality

$$(23.31) \qquad \| [\mathcal{P}]\Phi \| \leq \gamma \| (N + 2)\Phi \|$$

holds for all such states Φ. By the process of closure the field operator $[\mathcal{P}]$ can be extended into a domain $\mathfrak{H}_{\mathcal{P}}$ in which it is closed.

If the pseudo-operator $\mathcal{R}$ is pseudo-hermitian, in addition to being pseudo-antisymmetric, *the field operator* $[\mathcal{R}]$ *is antihermitian*, i.e.,

$$(23.32) \qquad [\mathcal{R}]^* = -[\mathcal{R}].$$

For, the hermitian adjoints of $A^+R_{--}A^-$, $A^+R_{-+}A^+$, $\mp A^-R_{+-}A^-$, $\mp A^-R_{++}A^+$ are respectively $A^{+\prime}\overline{R}_{--}A^- = -A^+R_{--}A^-$, $A^{-\prime}\overline{R}_{-+}A^- = \pm A^-R_{+-}A^-$, $\mp A^{+\prime}\overline{R}_{+-}A^+ = -A^+R_{-+}A^+$, $\mp A^{-\prime}\overline{R}_{++}A^+ = \pm A^-R_{++}A^+$. Moreover, *the operator* $i[\mathcal{R}]$ *is hypermaximal* (or self-adjoint) *in the space* $\mathfrak{H}_{\mathcal{R}}$ as could be readily shown.

First Commutator Identity

The justification of the first form of the canonical transformation, which we shall now set up, will be derived from the evaluation of the commutator of the two field operators $w \circ \mathcal{A}$ and $[\mathcal{P}] = \frac{1}{2}\, \mathcal{A} \circ \mathcal{P}\mathcal{A}$. This—proper—commutator is given by the identity

$$[w \circ \mathcal{A},\ [\mathcal{P}]] = w \circ \mathcal{P}\mathcal{A} \tag{23.33}$$

and is understood to hold when both sides are applied on a state which admits the operators $w \circ \mathcal{A}\,[\mathcal{P}]$ and $[\mathcal{P}]\, w \circ \mathcal{A}$; note that every state which admits the operator $N\sqrt{N}$ has this property. Symbolically, we may write identity (23.33) in the form

$$[\mathcal{A},\ [\mathcal{P}]] = \mathcal{P}\mathcal{A}. \tag{23.34}$$

In order to prove identity (23.33) we first employ identity (23.5) obtaining

$$(w \circ \mathcal{A})(\mathcal{A} \circ \mathcal{P}\mathcal{A}) = \pm \mathcal{A} \circ (w \circ \mathcal{A})\mathcal{P}\mathcal{A} + w \circ \mathcal{P}\mathcal{A};$$

next we employ relations (23.3)′ and (23.5)′ obtaining

$$\begin{aligned} \pm \mathcal{A} \circ (w \circ \mathcal{A})\mathcal{P}\mathcal{A} &= -\mathcal{A} \circ (\mathcal{A} \circ w)\mathcal{P}\mathcal{A} \\ &= -{}'\mathcal{P}\mathcal{A} \circ (\mathcal{A} \circ w)\mathcal{A} \\ &= \mp({}'\mathcal{P}\mathcal{A} \circ \mathcal{A})(\mathcal{A} \circ w) - w \circ {}'\mathcal{P}\mathcal{A} \\ &= (\mathcal{A} \circ \mathcal{P}\mathcal{A})(w \circ \mathcal{A}) - w \circ {}'\mathcal{P}\mathcal{A}. \end{aligned}$$

Addition of the two formulas yields

$$[w \circ \mathcal{A},\ (\mathcal{A} \circ \mathcal{P}\mathcal{A})] = w_0 \circ (\mathcal{P} - {}'\mathcal{P})\mathcal{A}$$

whence (23.33) because of ${}'\mathcal{P} = -\mathcal{P}$.

Exponential Function of Pseudo-biquantized Operators

The desired canonical transformation will be given with the aid of the exponential function $\exp\,[\mathcal{P}]$ of pseudo-biquantized operators $[\mathcal{P}]$. If the operator $[\mathcal{P}]$ is pseudo-hermitian so that $i\,[\mathcal{P}]$ is hypermaximal, the function $\exp\,[\mathcal{P}]$ is defined as a unitary operator according to the theory of hypermaximal operators, cf. [19], [25]. In order to define the operator $\exp\,[\mathcal{P}]$ without assuming that $[\mathcal{P}]$ is pseudo-hermitian one may expand the exponential function in a power series and apply it first to states Φ' with only a finite number of components in the particle representation. The resulting sequence of components does not necessarily define a state $\Phi'' = \exp\,[\mathcal{P}]\,\Phi'$ with a finite norm. This will be clear from results obtained later on, cf. (24.47), (24.48), in connection with the case in which only the term P_{-+} is different from zero, so that $[\mathcal{P}]$ involves only creation operators. It can be shown, however, that the series $\exp\,[\mathcal{P}]$ is applicable on states Φ' provided the norms of the operators $P_{\pm\pm}$ are small enough. The norm $\| P \|$ of an operator P acting on states Ψ of a single particle is the smallest number p such that the inequality $\| P\Psi \| \leq p \,\| \Psi \|$ holds for all states Ψ. After the operator $\exp\,[\mathcal{P}]$ has been defined for the states Φ' it can be extended by the process of closure to a dense subset of the Hilbert space of all states Φ.

The subsequent operations involving operators of the form $\exp[\mathcal{P}]$ could be justified—unless otherwise stated—provided the norms of the operators P_{**} are small enough. It remains to justify these operations assuming only that the operators $\exp[\mathcal{P}]$ involved are defined.

First Similarity Rule, First Form of the Operator T

For the field operator $\exp[\mathcal{P}]$ formed with the aid of the pseudo-biquantized operator $[\mathcal{P}]$ we can form the field operator $\exp[\mathcal{P}]$. For this operator we shall establish the identity

$$\exp[-\mathcal{P}](w \circ \mathcal{A}) \exp[\mathcal{P}] = w \circ (\exp \mathcal{P})\mathcal{A}, \tag{23.35}$$

valid for arbitrary functions $w(s)$. Since the left member of this formula represents a similarity transformation of the field operator $(w \circ \mathcal{A})$ we call this formula the first "similarity rule." Symbolically, identity (23.35) may be written in the form

$$\exp[-\mathcal{P}]\mathcal{A} \exp[\mathcal{P}] = (\exp \mathcal{P})\mathcal{A}. \tag{23.36}$$

If we specify $\mathcal{P}$ as the pseudo-operator $\mathcal{R}$ for which $(\exp \mathcal{R})\ \mathcal{A} = \mathcal{B}$ in accordance with (23.19), (23.17), we realize from (23.36) that the field operator

$$T_1 = \exp[\mathcal{R}] = \exp \tfrac{1}{2}\mathcal{A} \circ \mathcal{R}\mathcal{A} \tag{23.37}$$

satisfies the relation

$$\mathcal{A}T_1 = T_1\mathcal{B}, \tag{23.38}$$

which is the same as (22.34). Since the field operator $[\mathcal{R}]$ is anti-hermitian, cf. (23.32), the field operator T *is unitary* and can be extended to the whole Hilbert space $\mathfrak{H}$. Thus we see that *the field operator* $T_1 = \exp[\mathcal{R}]$ *gives the desired canonical transformation.*

We consider the field operator

$$I(t) = \exp[t\mathcal{P}]w \circ (\exp t\mathcal{P})\mathcal{A} \exp[-t\mathcal{P}],$$

and compute its derivative with respect to t formally; we obtain

$$\begin{aligned} &\exp[-t\mathcal{P}]I'(t) \exp[t\mathcal{P}] \\ &\qquad = [\mathcal{P}]w \circ \exp(t\mathcal{P})\mathcal{A} + w \circ (\exp t\mathcal{P})\mathcal{P}\mathcal{A} - w \circ (\exp t\mathcal{P})\mathcal{A}[\mathcal{P}] \\ &\qquad = \exp(t\mathcal{P})\{w' \circ \mathcal{P}\mathcal{A} - [(w' \circ \mathcal{A}), [\mathcal{P}]]\} \end{aligned}$$

with $w' = (\exp t\,'\mathcal{P})w$. According to identity (23.33), the operator in the curly brackets vanishes. Hence $I'(t) = 0$ and therefore $I(t) = I(0)$. Since, evidently, $I(0) = w \circ \mathcal{A}$ the relation $I(t) = w \circ \mathcal{A}$ is established.

24. *E-Ordering of the Canonical Transformation*

Specific information about the effects of the canonical transformation T cannot be derived directly from the expression $\exp \tfrac{1}{2}\ \mathcal{A} \circ \mathcal{R}\mathcal{A}$ for this transformation, see (23.37); for, creation and annihilation operators are still "interlocked" in this expression. The separation of these two types of operators will be performed in this section and various conclusions about the character of the transformation T will be drawn.

We shall use canonical transformations of the form

$$T = \exp[\mathcal{P}_1]\exp[\mathcal{P}_2]\cdots\exp[\mathcal{P}_n] \tag{24.1}$$

with appropriate pseudo-antisymmetric operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\cdots$. As an immediate consequence of the first similarity rule (23.36) we observe that for such a transformation the formula

$$T^{-1}\mathcal{A}T = \exp\mathcal{P}_1\exp\mathcal{P}_2\cdots\exp\mathcal{P}_n\mathcal{A} \tag{24.2}$$

holds. In order to ensure that the transformation T satisfies the desired condition

$$T^{-1}\mathcal{A}T = \mathcal{Y}\mathcal{A} = \mathcal{B} \tag{24.3}$$

it is, therefore, sufficient to ensure that the pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\cdots$, $\mathcal{P}_n$, satisfy the relation

$$\exp\mathcal{P}_1\exp\mathcal{P}_2\cdots\exp\mathcal{P}_n = \mathcal{Y}. \tag{24.4}$$

Special efforts, on the other hand, are needed to show that the operator T is unitary.

Second Commutator Identity

In deriving the decomposition (24.1) we shall use the identity

$$[[\mathcal{P}_1], [\mathcal{P}_2]] = [[\mathcal{P}_1, \mathcal{P}_2]] \tag{24.5}$$

which expresses the fact that the *commutator of two pseudo-biquantized operators is the pseudo-biquantization of their commutator.* This identity may also be written in the form

$$[\mathcal{A}\circ\mathcal{P}_1\mathcal{A}, \mathcal{A}\circ\mathcal{P}_2\mathcal{A}] = 2\mathcal{A}\circ[\mathcal{P}_1, \mathcal{P}_2]\mathcal{A}. \tag{24.5$_1$}$$

Of course, the identity is meant to be valid when applied to a state which admits the operators $[\mathcal{P}_1][\mathcal{P}_2]$, $[\mathcal{P}_2][\mathcal{P}_1]$, and $[[\mathcal{P}_1, \mathcal{P}_2]]$.

Identity (24.5) can be verified by deriving three identities from the first commutator identity (23.33): first

$$\begin{aligned}(\mathcal{A}\circ\mathcal{P}_1\mathcal{A})[\mathcal{P}_2] &= ({}'\mathcal{P}_1\mathcal{A}\circ\mathcal{A})[\mathcal{P}_2] \\ &= ({}'\mathcal{P}_1\mathcal{A}\circ[\mathcal{P}_2]\mathcal{A}) + {}'\mathcal{P}_1\mathcal{A}\circ\mathcal{P}_2\mathcal{A},\end{aligned}$$

then

$$\begin{aligned}({}'\mathcal{P}_1\mathcal{A}\circ[\mathcal{P}_2]\mathcal{A}) &= (\mathcal{A}\circ[\mathcal{P}_2]\mathcal{P}_1\mathcal{A}) \\ &= [\mathcal{P}_2](\mathcal{A}\circ\mathcal{P}_1\mathcal{A}) + \mathcal{P}_2\mathcal{A}\circ\mathcal{P}_1\mathcal{A},\end{aligned}$$

and finally, by addition,

$$[(\mathcal{A}\circ\mathcal{P}_1\mathcal{A}), [\mathcal{P}_2]] = \mathcal{A}\circ(\mathcal{P}_1\mathcal{P}_2 + {}'\mathcal{P}_2\mathcal{P}_1)\mathcal{A};$$

thus (24.5) follows because of $'\mathcal{P}_2 = -\mathcal{P}_2$. (Incidentally, the relation $'\mathcal{P}_1 = -\mathcal{P}_1$ has not been used here.)

Composition Rule

Our process of decomposition will be derived from the following fact: *Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_{12}$ be three pseudo-antisymmetric pseudo-operators which satisfy the relation*

$$\exp \mathcal{P}_1 \exp \mathcal{P}_2 = \exp \mathcal{P}_{12}\,; \tag{24.6}$$

then the corresponding pseudo-biquantized operators satisfy the corresponding relation

$$\exp\,[\mathcal{P}_1] \exp\,[\mathcal{P}_2] = \exp\,[\mathcal{P}_{12}]. \tag{24.7}$$

This formula will be referred to as the *composition rule for pseudo-biquantized operators.* We have not been able to obtain a complete proof for this rule; we are, therefore, forced to use it only heuristically. The second form of the canonical transformation T which we shall derive from it can, however, be verified independently.

It is possible to give a "formal" derivation of the composition rule, assuming that the exponential functions are formal power series. Multiplication of such formal series is to be performed term by term and rearrangement of the order of terms in a finite or infinite series is permitted but the order of multiplication can not be interchanged since the terms are not assumed to commute.

The term "Lie polynomial" of order $n > 1$ is used for every homogeneous polynomial in $\mathcal{P}_1$ and $\mathcal{P}_2$ which can be constructed from Lie polynomials of lower order solely by forming commutators and linear combinations with constant coefficients. A Lie polynomial of order 1 is any linear function of $\mathcal{P}_1$ and $\mathcal{P}_2$. Any formal series of Lie polynomials is called "Lie function" and is denoted by $l(\mathcal{P}_1, \mathcal{P}_2)$.

We can form Lie polynomials and Lie functions of pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$ as well as of field operators $[\mathcal{P}_1]$, $[\mathcal{P}_2]$. An immediate consequence of the second commutator identity (24.5) is that a Lie function $l\,([\mathcal{P}_1], [\mathcal{P}_2])$ of two pseudo-biquantized operators is the pseudo-biquantization $[l(\mathcal{P}_1, \mathcal{P}_2)]$ of the Lie function $l(\mathcal{P}_1, \mathcal{P}_2)$

$$l([\mathcal{P}_1], [\mathcal{P}_2]) = [l(\mathcal{P}_1, \mathcal{P}_2)]. \tag{24.8}$$

We make use of an important fact discovered independently by Baker and Hausdorff [1], [2] in 1905: *The function* log (exp $\mathcal{P}_1$) (exp $\mathcal{P}_2$) *is a Lie function.*[3]

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[3] Baker and Hausdorff [42,43] also gave a formula to describe this function explicitly in terms of commutators, but we do not need it here. For details see also a paper by W. Magnus [47].

The theorem of Baker and Hausdorff is an immediate consequence of the following characterization of Lie functions $f(\mathcal{P}_1, \mathcal{P}_2)$:

Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_1'$, $\mathcal{P}_2'$ be four elements such that both $\mathcal{P}_1$ and $\mathcal{P}_2$ commute with both $\mathcal{P}_1'$ and $\mathcal{P}_2'$; then

$$f(\mathcal{P}_1 + \mathcal{P}_1', \mathcal{P}_2 + \mathcal{P}_2') = f(\mathcal{P}_1, \mathcal{P}_2) + f(\mathcal{P}_1', \mathcal{P}_2').$$

For a proof of this fact see a forthcoming paper by Magnus in which various aspects and consequences of the theorem of Baker and Hausdorff are treated.

As a consequence of this theorem, formula (23.42) implies the formula

$$\log (\exp [\mathcal{P}_1])(\exp [\mathcal{P}_2]) = [\log (\exp \mathcal{P}_1)(\exp \mathcal{P}_2)].$$

Setting

$$\log (\exp \mathcal{P}_1)(\exp \mathcal{P}_2) = \mathcal{P}_{12}$$

in agreement with (24.6), we obtain

$$\log (\exp [\mathcal{P}_1])(\exp [\mathcal{P}_2]) = [\mathcal{P}_{12}]$$

and thus identity (24.7).

As a *Corollary to the composition rule* (24.7) we formulate: Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_{123}$ be four pseudo-antisymmetric pseudo-operators which satisfy the relation

$$(24.6)' \qquad \exp \mathcal{P}_1 \exp \mathcal{P}_2 \exp \mathcal{P}_3 = \exp \mathcal{P}_{123} ;$$

then the corresponding pseudo-biquantized operators satisfy the relation

$$(24.7)' \qquad \exp [\mathcal{P}_1] \exp [\mathcal{P}_2] \exp [\mathcal{P}_3] = \exp [\mathcal{P}_{123}].$$

This corollary is an immediate consequence of identity (24.7) whenever a pseudo-operator $\mathcal{P}_{12}$ exists such that (24.6) holds. Such a pseudo-operator evidently exists as a formal series and the corollary again follows formally from the Baker-Hausdorff Theorem.

Second Form of the Canonical Transformation

The first form (23.37) of the canonical transformation $T_1 = \exp [\mathcal{R}] = \exp \frac{1}{2}\mathcal{A} \circ \mathcal{R}\mathcal{A}$ does not yet allow one to carry out the transformation of the unmodified into the modified representers in a direct way. For, since annihilation and creation operators are still "interlocked" in this form, the determination of a modified component $\chi_m(s)_m$ contributed by an unmodified component $\psi_n(s)_n$ still requires the evaluation of an infinite series.

It is possible to cast the transformation T into a different form in which annihilation and creation operators are separated.

It is necessary to introduce three special types of pseudo-antisymmetric pseudo-operators. If the three coefficients E_{--}, E_{++}, E_{-+} vanish,

$$(24.9)_1 \qquad E_{--} = E_{++} = E_{-+} = 0,$$

for such a pseudo-operator we denote it by $\mathcal{E}$. We then set

$$(24.9)_2 \qquad E_{+-} = E,$$

so that

$$(24.9)_3 \qquad \mathcal{E} = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}.$$

By (23.22) we have

(24.10) $$[\mathcal{E}] = \tfrac{1}{2}\mathfrak{A} \circ \mathcal{E}\mathfrak{A} = \mp\tfrac{1}{2}A^{-}EA^{-}.$$

Note that this expression involves only annihilation operators.

We denote by $\mathcal{G}$ a pseudo-antisymmetric pseudo-operator for which

$(24.11)_1$ $$G_{--} = G_{++} = G_{+-} = 0,$$

and set

$(24.11)_2$ $$G_{-+} = G,$$

so that

$(24.11)_3$ $$\mathcal{G} = \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}.$$

Because of (23.22), the expression

(24.12) $$[\mathcal{G}] = \tfrac{1}{2}\mathfrak{A} \circ \mathcal{G}\mathfrak{A} = \tfrac{1}{2}A^{+}GA^{+}$$

involves only creation operators.

The condition that the pseudo-operators $\mathcal{E}$ and $\mathcal{G}$ be pseudo-antisymmetric is satisfied exactly if both E and G are symmetric for boson fields, antisymmetric for fermion fields, cf. $(23.14)_2$,

(24.13) $$'E = \pm E, \qquad 'G = \pm G.$$

We denote by $\mathfrak{F}$ a pseudo-antisymmetric pseudo-operator for which

$(24.14)_1$ $$F_{+-} = F_{-+} = 0$$

and set

$(24.14)_2$ $$F_{--} = F.$$

Because of $(23.14)_1$ we have

$(24.14)_3$ $$F_{++} = -'F;$$

hence

$(24.14)_4$ $$\mathfrak{F} = \begin{pmatrix} F & 0 \\ 0 & -'F \end{pmatrix}.$$

From (23.22) and (23.27) we derive

(24.15) $$[\mathfrak{F}] = \tfrac{1}{2}\mathfrak{A} \circ \mathfrak{F}\mathfrak{A} = A^{+}FA^{-} \pm \tfrac{1}{2}\,Tr\,F.$$

Thus, except for the additive constant $\pm\tfrac{1}{2}\,Tr\,F$, the pseudo-biquantized operator $[\mathfrak{F}]$ is exactly the proper biquantized operator $[F]$.

The *second form of the canonical trasnformation* can now be defined as

$$(24.16)_1 \qquad T_2 = \exp\,[\mathcal{E}]\exp\,[\mathcal{F}]\exp\,[\mathcal{G}]$$

or

$$(24.16)_2 \quad T_2 = \exp\,\{\mp\tfrac{1}{2}\,Tr\,F\}\exp\,\{\mp\tfrac{1}{2}A^-EA^-\}\exp\,\{A^+FA^-\}\exp\,\{\tfrac{1}{2}A^+GA^+\}.$$

The inverse operator T^{-1} is then of the form

$$(24.16)_3 \qquad \begin{aligned} T_2^{-1} = {} & \exp\,\{\pm\tfrac{1}{2}\,Tr\,F\} \\ & \cdot \exp\,\{-\tfrac{1}{2}A^+GA^+\}\exp\,\{-A^+FA^-\}\exp\,\{\pm\tfrac{1}{2}A^-EA^-\}. \end{aligned}$$

If an operator is given in this form it will be said to be *E-ordered*.[4] Here the quantification E is meant to refer to "exponential function."

In order to establish this form we must show that for every pseudo-antisymmetric pseudo-hermitian operator $\mathcal{R}$ there exist three pseudo-operators $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ of the type described such that the identity

$$(24.17) \qquad \exp\,[\mathcal{E}]\exp\,[\mathcal{F}]\exp\,[\mathcal{G}] = \exp\,[\mathcal{R}]$$

holds.

Such a decomposition of the field operator $\exp\,[\mathcal{R}]$ can now be simply effected by performing the corresponding decomposition for the pseudo-operator $\exp\,\mathcal{R}$ itself,

$$(24.18) \qquad \exp\,\mathcal{E}\exp\,\mathcal{F}\exp\,\mathcal{G} = \exp\,\mathcal{R}.$$

This fact is evidently implied by the Corollary to the Composition Rule formulated in the previous subsection, see (24.7)′. We shall show in a later subsection that the decomposition (24.18) can be effected very easily.

The advantage in using the second form of the transformation T becomes apparent if we desire to determine the N-representation of a state Φ whose M-representation is given. To this end we form the state Φ' whose N-representers are the same as the M-representers of Φ and set

$$(24.19) \qquad \Phi = T^{-1}\Phi' = \exp\,[-\mathcal{G}]\exp\,[-\mathcal{F}]\exp\,[-\mathcal{E}]\Phi'.$$

Let us first assume that Φ is an eigenstate of the modified number M with the eigenvalue m such that the only non-vanishing M-representer is $\chi_m(s)_m$. This function is then the only non-vanishing N-component of Φ'. We expand the exponential function $\exp[-\mathcal{E}]$ in a power series and apply each term on Φ'. Evidently, since $\mathcal{E}$ involves only annihilation operators, only a finite number, at most $m + 1$, terms of this series give a non-vanishing contribution. Since $\exp[-\mathcal{F}] = \exp\{-A^+FA^-\}\exp\{\mp \tfrac{1}{2}\,Tr\,F\}$, it is clear that the field operator $\exp[-\mathcal{F}]$ commutes with the number N and transforms the n-th component of $\exp[-\mathcal{E}]\,\Phi'$ into another n-particle component. The application of the field operator $\exp[-\mathcal{G}]$ produces components for every number n, but each such component consists of only a finite number—at most $m + 1$—terms. Thus the N-representers of the state Φ are easily calculated.

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[4] This E-ordering is different from the S-ordering, sometimes also called well-ordering, of an operator; see [49]. An S-ordered operator is given as an infinite series of which each term is a homogeneous polynomial in creation and annihilation operators, the annihilation operators acting first, and the creation operators afterwards. The series expansions of $\exp\,\{-[\mathcal{E}]\}$ and $\exp\,\{[\mathcal{G}]\}$ lead to S-ordered operators but not the expansion of $\exp\,\{-[\mathcal{F}]\}$. Since the latter operator commutes with the number $N = A^+A^-$, it is immediately applicable as it stands, cf. (25.40), (25.41), while derivation and application of its S-ordered form requires a few involved operations.

Modified Vacuum State

Let Φ be the modified vacuum state Φ^B_{vac} with the M-representers $\chi_0 = 1$, $\chi_m = 0$ for $m > 0$. Then the corresponding state Φ' is just the unmodified vacuum state Φ^A_{vac} and we may express Φ^B_{vac} in terms of Φ^A_{vac} through

$$\Phi^B_{\text{vac}} = T^{-1}\Phi^A_{\text{vac}}\,,$$

cf. (23.27). Evidently,

$$\exp[-\mathcal{E}]\Phi^A_{\text{vac}} = \Phi^A_{\text{vac}}$$

since all terms but the first in the series $\sum_{\nu=0}^{\infty} \nu\,\text{¡}\,[-\mathcal{E}]\,\nu$ transform Φ^A_{vac} into zero. For the same reason in view of (24.15) we have

$$\exp[-\mathcal{F}]\Phi^A_{\text{vac}} = \eta\Phi^A_{\text{vac}}$$

if we set

$$\eta = \exp\{\mp\tfrac{1}{2}\,Tr\,F\}. \tag{24.20}$$

Hence we are left with the relation

$$\Phi^B_{\text{vac}} = \eta\exp[-\mathcal{G}]\Phi^A_{\text{vac}}$$

or

$$\Phi^B_{\text{vac}} = \eta\exp\{-\tfrac{1}{2}A^+GA^+\}\Phi^A_{\text{vac}}\,. \tag{24.21}$$

A simple calculation now shows that the n-th representer of the state Φ^B_{vac} is

$$\psi^B_n(s)_n = \eta[n!]^{1/2}(n/2)\,\text{¡}\,(-2)^{-n/2}\binom{\text{Sy}}{\text{Asy}}g(s_n\,, s_{n-1})\cdots g(s_2\,, s_1), \tag{24.22}$$

$$\psi^B_0 = \eta,$$

provided n is even, and

$$\psi^B_n(s)_n = 0 \tag{24.23}$$

if n is odd.

In accordance with condition $(23.30)_2$ we must assume that the operator

G is square traceable so that G^S is an integral operator with a kernel $g(s', s'')$ for which

$$Tr\, GG^* = \iint |g(s', s'')|^2\, dm(s')\, dm(s'') < \infty. \tag{24.24}$$

The fact that the norm of the state Φ^B_{vac} is unity, which follows from the unitary character of T, is expressed by the relation

$$\begin{aligned} 1 + \sum_{\nu=1}^{\infty} (2\nu)!\,[\nu!]^2(-2)^{-2\nu} \int \left| \begin{pmatrix}\text{Sy}\\ \text{Asy}\end{pmatrix} g(s_{2\nu}, s_{2\nu-1}) \cdots g(s_2, s_1) \right|^2 dm(s)_{2\nu} \\ = |\eta|^{-2} = \exp\{\pm \Re e\, Tr\, F\}. \end{aligned} \tag{24.25}$$

Below, see (24.38), we shall derive the relation

$$\Re e\, Tr\, F = -\tfrac{1}{2}\, Tr \log(1 \mp GG^*),$$

which will enable us to evaluate the infinite series (24.25).

The significance of the number $|\eta|$ is, of course, clear from the statement that

$$|\eta|^2 = Pr\,(N, 0 \mid M, 0) \tag{24.26}$$

is *the probability that the field will be found in the unmodified vacuum state after it was found to be in the modified vacuum state.*

The probability that 2ν unmodified particles are found in the modified vacuum state is

$$\begin{aligned} Pr\,(N, 2\nu \mid M, 0) = \eta^2 (2\nu)!\,(\nu!)^2(-2)^{-2\nu} \\ \cdot \int \left| \begin{pmatrix}\text{Sy}\\ \text{Asy}\end{pmatrix} g(s_{2\nu}, s_{2\nu-1}) \cdots g(s_2, s_1) \right|^2 dm(s)_{2\nu}, \end{aligned} \tag{24.26$_1$}$$

as seen from (24.23). This expression can also be evaluated as the term of order ν in the expansion of $\exp\{\mp\tfrac{1}{2}\, Tr \log(1 \mp GG^*)\}$ with respect to powers of GG^*.

The possibility of expressing the transformation of the unmodified into the modified vacuum state with the aid of an operator $\exp\{-\tfrac{1}{2} A^+GA^+\}$ as in (24.21) was suggested by G. Goertzel.[5] Once this transformation is found, the transformation of the M-representers into the N-representers of any state Φ can be determined. One expresses the state Φ in terms of the state Φ^B_{vac} with the aid of the modified creation operators B^+, expressing these in terms of $A^{\pm}$, and Φ^B_{vac} in terms of Φ^A_{vac}. In this way the state Φ is obtained from Φ^A_{vac} with the aid of operators built up in terms of $A^{\pm}$. The N-representers of Φ can then be found.

In fact, it would not be difficult to determine the operator G solely from the condition

$$B^- \exp\{-\tfrac{1}{2}A^+GA^+\}\Phi^A_{vac} = 0$$

and to proceed as indicated.

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[5] In connection with Sollfrey's thesis [50]. This suggestion was the starting point for the investigations presented in this Part V.

Nevertheless, we prefer to present our procedure of E-ordering because it leads to easily managed explicit formulas.

First Decomposition

In order to effect the decomposition (24.18) of the pseduo-operator exp $\mathcal{R}$, we first note that the pseudo-operators exp $\mathcal{E}$ and exp $\mathcal{G}$ reduce to

$$\begin{aligned} \exp \mathcal{E} &= 1 + \mathcal{E}, \\ \exp \mathcal{G} &= 1 + \mathcal{G}; \end{aligned} \tag{24.27}$$

for, as seen from (24.9) and (24.11), $\mathcal{E}^2 = 0$ and $\mathcal{G}^2 = 0$. Explicitly we have

$$\begin{aligned} \exp \mathcal{E} &= \begin{pmatrix} 1 & 0 \\ E & 1 \end{pmatrix}, \\ \exp \mathcal{G} &= \begin{pmatrix} 1 & G \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag*{(24.27)$_1$}$$

and

$$\exp \mathcal{F} = \begin{pmatrix} \exp F & 0 \\ 0 & \exp(-'F) \end{pmatrix}, \tag{24.28}$$

according to (24.14). A simple calculation then gives

$$\exp \mathcal{E} \exp \mathcal{F} \exp \mathcal{G} = \begin{pmatrix} \exp F & (\exp F)G \\ E \exp F & E(\exp F)G + \exp(-'F) \end{pmatrix}$$

and comparison with

$$\exp \mathcal{R} = \mathcal{Y} = \begin{pmatrix} Y_{--} & Y_{-+} \\ Y_{+-} & Y_{++} \end{pmatrix}$$

yields the relations

$$\begin{aligned} \exp F &= Y_{--}, \\ (\exp F)G &= Y_{-+}, \\ E \exp F &= Y_{+-}, \\ E \exp FG + \exp(-'F) &= Y_{++}. \end{aligned} \tag{24.29}$$

Thus the operator F is determined as

$$F = \log Y_{--}, \tag{24.30}$$

and G and E as

(24.31)$_1$ $$G = Y_{--}^{-1} Y_{-+} ,$$

(24.31)$_2$ $$E = Y_{+-} Y_{--}^{-1} .$$

The last relation (24.29) is automatically satisfied; for, as will be seen, it is a consequence of the conditions $\mathcal{Y}'\mathcal{Y} = 1$ and $'\mathcal{Y}\mathcal{Y} = 1$. The latter conditions also insure that E and G satisfy the symmetry conditions (24.13). In fact, relation

$$\mp Y_{--}{}'Y_{-+} + Y_{-+}{}'Y_{--} = 0$$

implied by $\mathcal{Y}'\mathcal{Y} = 1$, cf. (24.9), yields $'G = \pm G$, and the relation

$$\mp{}'Y_{+-}Y_{--} + {}'Y_{--}Y_{+-} = 0$$

implied by $'\mathcal{Y}\mathcal{Y} = 1$ yields $'E = \pm E$. Setting $G = \pm{}'G$, in the last relation (24.29) we may write it in the form

$$\pm Y_{+-}{}'Y_{-+}({}'Y_{--})^{-1} + ({}'Y_{--})^{-1} = Y_{++} ;$$

hence this relation follows from the relation

$$Y_{++}{}'Y_{--} \mp Y_{+-}{}'Y_{-+} = 1$$

implied by $\mathcal{Y}'\mathcal{Y} = 1$.

With these considerations we have derived the second, E-ordered, form (24.16) of the canonical transformation T^{-1} from the Composition Rule, provided the operator log Y_{--} can be formed. It is worth noting that the single particle operators E, F, G entering this second form are expressed directly in terms of the operators $Y_{\pm\pm}$ and not in terms of the pseudo-operator $\mathcal{R} = \log \mathcal{Y}$. Thus the second form of T can be described independently of the assumption that the pseudo-operator $\mathcal{Y}$ can be written in the form $\mathcal{Y} = \exp \mathcal{R}$.

Since we have not proved the Composition Rule rigorously, we should verify directly that the operator T_2 has the desired properties. It was already stated, cf. (24.3), that the relation

$$T_2^{-1}\mathcal{A}T_2 = \mathcal{B}$$

is a consequence of the first similarity rule. It is apparently not so simple to prove the unitary character of T_2 ; however we shall do so in Section 25.

Relations between E, F, G, and $\mathcal{Y}$

Since various expressions will be needed later for the operators E and G in terms of the coefficients of $\mathcal{Y}$, they will be enumerated here:

(24.32) $$G = Y_{--}^{-1} Y_{-+} = \pm{}'Y_{-+}{}'Y_{--}^{-1}$$

(24.33) $$\overline{G} = Y_{++}^{-1} Y_{+-} = \pm{}'Y_{+-}{}'Y_{++}^{-1}$$

(24.34) $$E = Y_{+-} Y_{--}^{-1} = \pm{}'Y_{--}^{-1}{}'Y_{+-}$$

(24.35) $$\overline{E} = Y_{-+} Y_{++}^{-1} = \pm{}'Y_{++}^{-1}{}'Y_{-+} .$$

The first column is implied by (22.31); the second column follows from the symmetry property of E and G established above. Furthermore, we have

$$1 - G\overline{G} = ('Y_{++}Y_{--})^{-1} = Y_{--}^{-1}\,'Y_{++}^{-1}\,, \tag{24.36}$$

$$1 - \overline{G}G = ('Y_{--}Y_{++})^{-1} = Y_{++}^{-1}\,'Y_{--}^{-1}\,, \tag{24.37}$$

$$1 - E\overline{E} = (Y_{++}\,'Y_{--})^{-1} = {}'Y_{--}^{-1}Y_{++}^{-1}\,, \tag{24.38}$$

$$1 - \overline{E}E = (Y_{--}\,'Y_{++})^{-1} = {}'Y_{++}^{-1}Y_{--}^{-1}\,. \tag{24.39}$$

The first relation follows from

$$Y_{--}[1 - G\overline{G}]'Y_{++} = Y_{--}\,'Y_{++} \mp Y_{-+}\,'Y_{+-} = 1$$

by (24.32), (24.33) and $\mathcal{Y}'\underline{\mathcal{Y}} = \underline{1}$. The other relations are derived similarly.

Since the operators $\pm G = {}'\overline{G} = G^*$ and $\pm\,\overline{E} = {}'\overline{E} = E^*$ are the Hermitian adjoints of the operators G and E, it is clear that the operators $1 - G\,\overline{G}$, $1 - \overline{G}\,G$, $1 - E\overline{E}$, $1 - \overline{E}E$ are non-negative and hypermaximal. Therefore these operators possess non-negative square roots. For example, we may introduce a non-negative operator ${}_1F$ through the relation

$$\exp {}_1F = ['Y_{++}Y_{--}]^{1/2} = [1 - G\overline{G}]^{-1/2}. \tag{24.40$_1$}$$

We may also introduce an anti-hermitian operator ${}_0F$ by means of

$$\exp {}_0F = Y_{--}['Y_{++}Y_{--}]^{1/2}. \tag{24.40$_2$}$$

Evidently,

$$\exp {}_0F^* = ['Y_{++}Y_{--}]^{-1/2}\,'Y_{++} = ['Y_{++}Y_{--}]^{1/2}Y_{--}^{-1} = \exp\{-{}_0F\}.$$

Consequently, the operator $i\,{}_0F$ is hermitian, in fact it is hypermaximal. Because of

$$\exp F = Y_{--}\,, \tag{24.29$_1$}$$

relation

$$\exp {}_0F \exp {}_1F = \exp F \tag{24.41}$$

holds.

Trace Relations

We maintain that relation (24.41) implies relation

$$\exp Tr\,{}_0F \exp Tr\,{}_1F = \exp Tr\,F. \tag{24.42}$$

This *trace composition rule*[6] is a—rather simple—analogue of the general composition rule (24.7). Like the general composition rule, it follows from the Baker-Hausdorff Theorem; for, the trace of a commutator of two operators is the commutator of the traces of these operators, namely zero.

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[6] If the operator F acts in a finite dimensional space the expression exp Tr log F reduces to the determinant of the matrix F^S. The trace composition rule (24.42) is, therefore, the analogue of the rule that the determinant of the product of two matrices is the product of their determinants.

Since the operator $i\ {}_0F$ is hermitian its trace is real; i.e., $Tr\ {}_0F$ is imaginary. Consequently, $|\exp Tr\ {}_0F| = 1$ so that

$$|\exp Tr\ F| = \exp Tr\ {}_1F \tag{24.43}$$

or, by $(24.40)_1$,

$$\mathcal{R}e\ Tr\ F = Tr\ {}_1F = -\tfrac{1}{2}\ Tr \log [1 - G\overline{G}]. \tag{24.44}$$

The absolute value of the number η introduced earlier, see (24.20), can now be expressed as

$$|\eta| = \exp \{\mp\tfrac{1}{2}\ Tr\ {}_1F\}$$

or, by $(24.40)_1$ and (24.44), as

$$\begin{aligned} |\eta| &= \exp \{\mp\tfrac{1}{4}\ Tr \log {}'Y_{++}Y_{--}\} \\ &= \exp \{\pm\tfrac{1}{4}\ Tr \log [1 - G\overline{G}]\}. \end{aligned} \tag{24.45}$$

Since, obviously, the operator $\mp\log [1 - G\overline{G}] = \mp\log [1 \mp GG^*]$ is positive definite, its trace is positive, and consequently $|\eta| < 1$, in agreement with (24.25). Incidentally, formula (24.45) gives an evaluation of the infinite series (24.25),

$$\begin{aligned} 1 + \sum_{\nu=1}^{\infty} (2\nu!)[\nu!]^2(-2)^{2\nu} \\ \cdot \int \left| \begin{pmatrix}\text{Sy}\\ \text{Asy}\end{pmatrix} g(s_{2\nu}, s_{2\nu-1}) \cdots g(s_2, s_1) \right|^2 dm(s)_{2\nu} \\ = \exp \frac{1}{2} \sum_{\nu=1}^{\infty} (\pm 1)^{\nu-1}\nu^{-1} \\ \cdot \int g(s_1, s_{2\nu})\overline{g(s_{2\nu}, s_{2\nu-1})} \cdots g(s_3, s_2)\overline{g(s_2, s_1)}\ dm(s)_{2\nu}, \end{aligned} \tag{24.46}$$

which could perhaps be derived directly.[7]

Conditions for the Existence of the Canonical Transformation

In conclusion, we summarize the conditions under which the second form of the canonical transformation $T_2 = \exp [\mathcal{E}] \exp [\mathcal{F}] \exp [\mathcal{G}]$ exists.

We have assumed that the operators $Y_{\pm\pm}$ are bounded and that the pseudo-operator $\mathcal{Y}$ is pseudo-hermitian and satisfies the conditions $\mathcal{Y}'\mathcal{Y} = {}'\mathcal{Y}\mathcal{Y} = 1$;

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[7] This formula is related to the evaluation of an infinite series derived by Feynman [4] with the aid of his graphs.

moreover, we have tacitly assumed that the operators Y_{--} and Y_{++} possess inverses. As a consequence, relation (24.36) holds and, since Y^*_{--} and Y_{--} are bounded, the operator $1 \mp G\,G^*$ has a positive lower bound. This fact may be expressed by the relation

$$|| GG^* || < 1 \tag{24.47}$$

in the boson case, if we denote by $|| GG^* ||$ the least upper bound of $(\psi, GG^*\psi) = || G^* \psi ||^2$ for all functions ψ in $\mathfrak{H}$ with $|| \psi || = 1$; similarly, $|| E^*E || < 1$ holds in the boson case. For the existence of the operators $[\mathcal{G}]$ and $[\mathcal{E}]$ additional restrictions must be imposed, namely that the operators G and E possess square traces:

$$Tr\, GG^* < \infty \quad , \quad Tr\, E^*E < \infty . \tag{24.48}$$

We maintain that under this condition combined with (24.47) the operator $\mp \log\,[1 \mp GG^*]$ also possesses a finite trace. This can be deduced most easily from the fact that the condition $Tr\; GG^* < \infty$ implies that the operator GG^* has a pure point spectrum of eigenvalues $\gamma^2 \geq 0$ for which $\sum \gamma^2 = Tr\; GG^*$. As a consequence it is seen that

$$Tr \log\,[1 + GG^*] = \sum \log\,[1 + \gamma^2] \leq \sum \gamma^2;$$

hence we may conclude that

$$Tr \log\,[1 + GG^*] \leq Tr\, GG^* \tag{24.49$_-$}$$

in the fermion case. Furthermore, we have

$$-Tr \log\,[1 - GG^*] = -\sum \log\,[1 - \gamma^2] \leq [1 - \gamma^2_{\max}]^{-1} \sum \gamma^2,$$

and since $\gamma^2_{\max} = || GG^* ||$, we may conclude that

$$-Tr \log\,[1 - GG^*] \leq [1 - || GG^* ||]^{-1}\, Tr\, GG^* \tag{24.49$_+$}$$

in the boson case.

The conditions (24.47), (24.48) are somewhat stronger than necessary for the existence of the transformation T_2 ; however, the conditions $Tr\; GG^* < \infty$ and $Tr\; E^*E < \infty$ are necessary.

If these conditions are not satisfied, the field does not possess a particle representation with respect to the operators $B^{\mp}$ as creation and annihilation operators. Nevertheless, it would seem likely that the field possesses an occupation number representation of the second type in the sense explained in Section 19, Part IV. The total weight associated with this representation would then be

$$W = \mp\tfrac{1}{2}\, Tr \log\,[1 \mp GG^*] \tag{24.50}$$

and hence infinite if $Tr\, GG^* = \infty$; in other words, if $T_2 GG^* = \infty$ the field would be "myriotic" with respect to the operators $B^{\mp}$. Although it would be of interest to follow up this possibility, it will not be done here.

25. *Third and Fourth Form of the Canonical Transformation*

In this section we shall establish two further forms of the transformation T. The third form involves only the operators G, ${}_0F$ and their conjugates; the fourth form results from the third by E-ordering. It is possible to show that the fourth form is identical with the third as well as with the second without using the composition rule. The third form is evidently unitary and thus it follows that the others are also unitary.

The main reason for discussing these other forms is not so much that we want to prove the identity of the first and second forms of the operator T. It is rather that we want to have the opportunity to formulate and apply other similarity and decomposition rules. The content of the present Section 25 is not essentially used in the subsequent sections.

Third and Fourth Form of the Transformation T

The third form of the canonical transformation will be derived from a "second" decomposition

$$\mathcal{Y} = \exp {}_0\mathfrak{F} \exp {}_1\mathfrak{R} \tag{25.1}$$

of the pseudo-operator $\mathcal{Y} = \exp \mathfrak{R}$ in which the pseudo-operator

$$ {}_0\mathfrak{F} = \begin{pmatrix} {}_0F & 0 \\ 0 & {}_0\overline{F} \end{pmatrix} \tag{25.2}$$

possesses only diagonal terms while terms of this type are absent from the pseudo-operator

$$ {}_1\mathfrak{R} = \begin{pmatrix} 0 & {}_1R_{-+} \\ {}_1R_{+-} & 0 \end{pmatrix}. \tag*{(25.2)$_1$}$$

These pseudo-operators ${}_0\mathfrak{F}$ and ${}_1\mathfrak{R}$, which we shall describe below, will be seen to be pseudo-antisymmetric and pseudo-hermitian, cf. (23.14) and (23.16). Having found the decomposition (25.1) we shall obtain from the composition identity (24.7) the *third form*

$$T_3 = \exp [{}_0\mathfrak{F}] \exp [{}_1\mathfrak{R}] \tag{25.3}$$

of the canonical transformation T in which, by (23.22), (23.27),

$$\exp [{}_0\mathfrak{F}] = \exp \{\mp\tfrac{1}{2}\, Tr\, {}_0F\} \exp \{A^+\, {}_0F A^-\} \tag{25.4}$$

and

$$\exp [{}_1\mathfrak{R}] = \exp \{\tfrac{1}{2}A^+\, {}_1R_{-+}\, A^+ \mp \tfrac{1}{2}A^-\, {}_1R_{+-}\, A^-\}. \tag*{(25.4)$_1$}$$

Because of the pseudo-hermitian character of ${}_0\mathfrak{F}$ and ${}_1\mathfrak{R}$, both operators $\exp [{}_0\mathfrak{F}]$

and $\exp [{}_1\mathcal{R}]$ are unitary; hence it is evident that the operator $\exp [{}_0\mathcal{F}] \exp [{}_1\mathcal{R}]$ is unitary.

The fourth form of the operator T will be obtained by decomposing the pseudo-operator $\exp {}_1\mathcal{R}$ as

$$\exp {}_1\mathcal{R} = \exp {}_1\mathcal{E} \exp {}_1\mathcal{F} \exp {}_1\mathcal{G} \tag{25.5}$$

corresponding to the first decomposition (24.18). The composition identity, cf. (24.17), then gives the *fourth form*

$$T_4 = \exp [{}_0\mathcal{F}] \exp [{}_1\mathcal{E}] \exp [{}_1\mathcal{F}] \exp [{}_1\mathcal{G}] \tag{25.6}$$

of the canonical transformation, where, as in the second form (24.16), annihilation and creation operators are separated.

We shall show that the identity of the fourth form of T with both the third and second forms can be verified without using the composition identity—which was not proved, but only derived formally. Since the third form is evidently unitary the same is true of the E-ordered second and fourth forms once their identity is established. Note that the pseudo-operators $\mathcal{E}$, $\mathcal{G}$ and ${}_1\mathcal{E}$, ${}_1\mathcal{G}$ are—in general—not pseudo-hermitation and that therefore—in general—the operators $\exp [\mathcal{E}]$, $\exp [\mathcal{G}]$ and $\exp [{}_1\mathcal{E}]$, $\exp [{}_1\mathcal{G}]$ are not unitary.

Various other advantages of the fourth form will be discussed later on.

Second Decomposition

In order to effect the second decomposition (25.1) of the pseudo-operator

$$\mathcal{Y} = \begin{pmatrix} Y_{--} & Y_{-+} \\ Y_{+-} & Y_{++} \end{pmatrix},$$

see (23.6), we first decompose it in the form

$$\mathcal{Y} = {}_0\mathcal{Y}\, {}_1\mathcal{Y} \tag{25.7}$$

with

$$ {}_0\mathcal{Y} = \begin{pmatrix} {}'Y_{++}^{-1}[{}'Y_{++}Y_{--}]^{1/2} & 0 \\ 0 & {}'Y_{--}^{-1}[{}'Y_{--}Y_{++}]^{1/2} \end{pmatrix} \tag{25.8$_0$}$$

and

$$ {}_1\mathcal{Y} = \begin{pmatrix} [{}'Y_{++}Y_{--}]^{1/2} & [{}'Y_{++}Y_{--}]^{1/2}Y_{--}^{-1}Y_{-+} \\ [{}'Y_{--}Y_{++}]^{1/2}Y_{++}^{-1}Y_{+-} & [{}'Y_{--}Y_{++}]^{1/2} \end{pmatrix}. \tag{25.8$_1$}$$

The two diagonal terms in the pseudo-operator ${}_0\mathcal{Y}$ are evidently unitary since $\overline{Y}_{--} = Y_{++}$, ${}'\overline{Y}_{++} = {}'Y_{--}$. The first of these terms is in fact identical with the right member of relation $(24.40)_2$. The two terms can therefore be written in the form $\exp {}_0F$ and $\exp {}_0\overline{F}$ with ${}_0F = \log \{Y_{--}[{}'Y_{++}Y_{--}]^{1/2}\}$. Consequently, the pseudo-operator ${}_0\mathfrak{Y}$ is of the form

$$ {}_0\mathfrak{Y} = \exp {}_0\mathfrak{F}, \tag{25.9} $$

where the pseudo-operator ${}_0\mathfrak{F}$ is given by (25.2) and $(24.40)_2$.

Evidently, the pseudo-operator ${}_0\mathfrak{F}$ is pseudo-hermitian and pseudo-antisymmetric and hence ${}_0\mathfrak{Y}$ is unitary.

The elements of the pseudo-operator ${}_1\mathfrak{Y}$ can easily be expressed in terms of the operators

$$ G = Y_{--}^{-1} Y_{-+} = \pm' Y_{-+}{}' Y_{--}^{-1} \tag*{(25.10)$_-$} $$

and

$$ \bar{G} = Y_{++}^{-1} Y_{+-} = \pm' Y_{+-}{}' Y_{++}^{-1}\,, \tag*{(25.10)$_+$} $$

cf. (23.32), (23.33). In fact, using the formulas

$$ 'Y_{++} Y_{--} = [1 - G\bar{G}]^{-1/2}, \qquad 'Y_{--} Y_{++} = [1 - \bar{G}G]^{-1/2}, \tag{25.11} $$

cf. (24.36), (24.37), and noting that $G^* = \pm\bar{G}$ we may write

$$ {}_1\mathfrak{Y} = \begin{pmatrix} [1 - G\bar{G}]^{-1/2} & [1 - G\bar{G}]^{-1/2} G \\ [1 - \bar{G}G]^{-1/2}\bar{G} & [1 - \bar{G}G]^{-1/2} \end{pmatrix}. \tag{25.12} $$

We now maintain that his pseudo-operator ${}_1\mathfrak{Y}$ is of the form

$$ {}_1\mathfrak{Y} = \exp {}_1\mathfrak{R} \tag{25.13} $$

if we choose for ${}_1\mathfrak{R}$ the pseudo-operator

$$ {}_1\mathfrak{R} = \begin{pmatrix} 0 & h(G\bar{G})G \\ h(\bar{G}G)\bar{G} & 0 \end{pmatrix}, \tag{25.14} $$

with

$$ h(z) = z^{-1/2} \operatorname{arctanh} z^{1/2}. $$

This statement may be proved by a power series expansion of $\cosh {}_1\mathfrak{R}$ and $\sinh {}_1\mathfrak{R}$, $\operatorname{arctanh} z^{1/2}$ making use of the identities

$$ \bar{G} f(G\bar{G}) = f(\bar{G}G)\bar{G}, \tag*{(25.15)$_1$} $$

$$ f(G\bar{G})G = G f(\bar{G}G), \tag*{(25.15)$_2$} $$

which are evident for polynomials f and then follow for functions f for which the operators $f(G\bar{G})$ and $f(\bar{G}G)$ are defined. Thus the second decomposition of $\mathfrak{Y}$ is effected.

The pseudo-operators ${}_1\mathfrak{E}$, ${}_1\mathfrak{F}$, ${}_1\mathfrak{G}$ of the final decomposition (25.5) can be obtained from the terms of the pseudo-operator ${}_1\mathfrak{Y}$ as given by (25.12) with the aid of the formulas (24.30), (24.31) by substituting the terms of ${}_1\mathfrak{Y}$ instead of those of $\mathcal{Y}$. We find

$$ {}_1F = -\tfrac{1}{2}\log[1 - G\bar{G}] \tag{25.16} $$

$$ {}_1G = G, \qquad {}_1E = \bar{G}; \tag{25.17} $$

in deriving (25.17), identity $(25.15)_1$ is to be used. The pseudo-operators ${}_1\mathcal{G}$, ${}_1\mathcal{E}$, ${}_1\mathcal{F}$ are then

$$ {}_1\mathcal{G} = \begin{pmatrix} 0 & G \\ 0 & 1 \end{pmatrix}, \quad {}_1\mathcal{E} = \begin{pmatrix} 0 & 0 \\ \bar{G} & 0 \end{pmatrix}, \tag{25.18} $$

$$ {}_1\mathcal{F} = -\frac{1}{2}\begin{pmatrix} \log[1 - G\bar{G}] & 0 \\ 0 & -\log[1 - \bar{G}G] \end{pmatrix} \tag{25.19} $$

and their exponential functions are

$$ \exp {}_1\mathcal{G} = \begin{pmatrix} 1 & G \\ 0 & 1 \end{pmatrix}, \quad \exp {}_1\mathcal{E} = \begin{pmatrix} 1 & 0 \\ \bar{G} & 1 \end{pmatrix}, \tag{25.20} $$

$$ \exp {}_1\mathcal{F} = \begin{pmatrix} [1 - G\bar{G}]^{-1/2} & 0 \\ 0 & [1 - \bar{G}G]^{1/2} \end{pmatrix}. \tag{25.21} $$

Final Form of the Transformation T

The fourth form of the transformation T can now finally be written as

$$ \begin{aligned} T_4 = {} & \exp\{\mp\tfrac{1}{2}\,Tr\,{}_0F\}\exp\{\pm\tfrac{1}{2}\,Tr\log[1 - \bar{G}G]\}\exp\{A^+\,{}_0FA^-\} \\ & \cdot\exp\{\mp\tfrac{1}{2}A^-\bar{G}A^-\}\exp\{A^+\log[1 - G\bar{G}]^{1/2}A^-\}\exp\{\tfrac{1}{2}A^+GA^+\}. \end{aligned} \tag{25.22} $$

Since the operator ${}_0F$ is anti-hermitian the first factor here is a number of absolute value 1 and therefore, rather insignificant. The second factor, on the other hand, is a positive number less than one. Actually, this factor is simply the absolute value of the number η introduced above, see (24.45).

Identification of the Third and Fourth Form of T

In order to prove the identity of the third and fourth forms of the canonical transformation, we replace G and $\bar{G}$ by tG and $t\bar{G}$ in ${}_1\mathcal{E}$, ${}_1\mathcal{G}$, ${}_1\mathcal{F}$ as given by (25.18), (25.19), letting t be a real parameter. We note that the derivatives of ${}_1\mathcal{E}$, ${}_1\mathcal{G}$, ${}_1\mathcal{F}$ with respect to t are given by

$$ {}_1\dot{\mathcal{E}} = t^{-1}\,{}_1\mathcal{E}, \qquad {}_1\dot{\mathcal{G}} = t^{-1}\,{}_1\mathcal{G}, \tag{25.23} $$

$$ {}_1\dot{\mathcal{F}} = \begin{pmatrix} [1 - t^2G\bar{G}]tG\bar{G} & 0 \\ 0 & -[1 - t^2\bar{G}G]t\bar{G}G \end{pmatrix}. \tag{25.24} $$

Evidently, each of the pseudo-operators ${}_1\mathcal{E}$, ${}_1\mathcal{G}$, ${}_1\mathcal{F}$ commutes with its derivative. The derivatives of the exponential functions are therefore given by

$$\frac{d}{dt}\exp{}_1\mathcal{E} = {}_1\dot{\mathcal{E}}\exp{}_1\mathcal{E}, \qquad \frac{d}{dt}\exp{}_1\mathcal{G} = {}_1\dot{\mathcal{G}}\exp{}_1\mathcal{G}, \tag{25.25}$$

$$\frac{d}{dt}\exp{}_1\mathcal{F} = {}_1\dot{\mathcal{F}}\exp{}_1\mathcal{F}. \tag{25.26}$$

We observe that the derivative

$${}_1\dot{\mathcal{R}} = \begin{pmatrix} 0 & [1 - t^2G\overline{G}]^{-1}G \\ [1 - t^2\overline{G}G]^{-1}\overline{G} & 0 \end{pmatrix} \tag{25.27}$$

of the pseudo-operator ${}_1\mathcal{R}$, given by (25.14), also commutes with ${}_1\mathcal{R}$ so that

$$\frac{d}{dt}\exp{}_1\mathcal{R} = {}_1\dot{\mathcal{R}}\exp{}_1\mathcal{R}.$$

Since, by (25.5)

$$\exp{}_1\mathcal{E}\exp{}_1\mathcal{F}\exp{}_1\mathcal{G}\exp(-{}_1\mathcal{R}) = 1,$$

the derivative of this pseudo-operator with respect to t vanishes. Hence we obtain the relation

$$\begin{aligned} &\exp(-{}_1\mathcal{G})\exp(-{}_1\mathcal{F}){}_1\dot{\mathcal{E}}\exp{}_1\mathcal{F}\exp{}_1\mathcal{G} \\ &\qquad\qquad + \exp(-{}_1\mathcal{G}){}_1\dot{\mathcal{F}}\exp{}_1\mathcal{G} + {}_1\dot{\mathcal{G}} - {}_1\dot{\mathcal{R}} = 0, \end{aligned} \tag{25.28}$$

which, of course, could also be verified directly.

Now we consider the operator

$$K = \exp[{}_1\mathcal{E}]\exp[{}_1\mathcal{F}]\exp[{}_1\mathcal{G}]\exp[-{}_1\mathcal{R}];$$

our aim is to prove that it is the identity. Again we replace G and $\overline{G}$ by tG and $t\overline{G}$ and differentiate with respect to t. As before, the drivative of each of the exponents $[{}_1\mathcal{E}], \cdots, [{}_1\mathcal{R}]$ commutes with the exponent. Consequently, the derivative $\dot{K}$ of K is given by

$$\dot{K} = \exp[{}_1\mathcal{E}]\exp[{}_1\mathcal{F}]\exp[{}_1\mathcal{G}]Z\exp[-{}_1\mathcal{R}]$$

with

$$\begin{aligned} Z = {} &\exp[-{}_1\mathcal{G}]\exp[-{}_1\mathcal{F}][{}_1\dot{\mathcal{E}}]\exp[{}_1\mathcal{F}]\exp[{}_1\mathcal{G}] \\ &\qquad\qquad + \exp[-{}_1\mathcal{G}][{}_1\dot{\mathcal{F}}]\exp[{}_1\mathcal{G}] + [{}_1\dot{\mathcal{G}}] - [{}_1\dot{\mathcal{R}}]. \end{aligned}$$

Now we make use of the "second similarity rule" which we shall prove in the next subsection. It states that the relation

$$\exp(-\mathcal{P}_1)\mathcal{P}_2\exp\mathcal{P}_1 = \mathcal{P}_3$$

between pseudo-antisymmetric pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ entails the relation

$$\exp\,[-\mathcal{P}_1][\mathcal{P}_2]\exp\,[\mathcal{P}_1] = [\mathcal{P}_3]$$

for the corresponding pseudo-biquantized operators. From this rule in conjunction with relation (25.28) we immediately infer

$$Z = 0$$

and hence $\dot{K} = 0$. Consequently, K is constant, and since K reduces to the identity for $t = 0$, the desired statement $K = 1$ follows. Thus the identity of the third and the fourth forms of the operator T is established.

Second and Third Similarity Rule

The *second similarity rule* which states that the relation

$$\exp\,(-\mathcal{P}_1)\mathcal{P}_2\exp\mathcal{P}_1 = \mathcal{P}_3 \tag{25.29}$$

between three pseudo-antisymmetric operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ entails the relation

$$\exp\,[-\mathcal{P}_1][\mathcal{P}_2]\exp\,[\mathcal{P}_1] = [\mathcal{P}_3] \tag{25.30}$$

between the corresponding biquantized operators can be proved in the same way as the first similarity rule (23.35). One replaces $\mathcal{P}_1$ by $t\mathcal{P}_1$ and differentiates the operator $J = \exp\,[t\mathcal{P}_1][\exp\,(-t\mathcal{P}_1)\mathcal{P}_2\exp\,[t\mathcal{P}_1]\exp\,[-t\mathcal{P}_1]$ with respect to t, obtaining

$$\begin{aligned}\exp\,[t\mathcal{P}_1]\{[[\mathcal{P}_1],\ &[\exp\,(-t\mathcal{P}_1)\mathcal{P}_2\exp t\mathcal{P}_1]]\\ &- [\exp\,(-t\mathcal{P}_1)[\mathcal{P}_1\,,\,\mathcal{P}_2]\exp t\mathcal{P}_1]\}\exp\,[-t\mathcal{P}_1].\end{aligned}$$

This expression vanishes since, by the second commutator rule (24.5),

$$\begin{aligned}[[\mathcal{P}_1],\ [\exp\,(-t\mathcal{P}_1)\mathcal{P}_2\exp t\mathcal{P}_1]] &= [[\mathcal{P}_1\,,\,\exp\,(-t\mathcal{P}_1)\mathcal{P}_2\exp t\mathcal{P}_1]]\\ &= [\exp\,(-t\mathcal{P}_1)[\mathcal{P}_1\,,\,\mathcal{P}_2]\exp t\mathcal{P}_1].\end{aligned}$$

Consequently, the operator J is independent of t and hence equal to $[\mathcal{P}_2]$ since it reduces to this operator for $t = 0$. In order to make this reasoning complete, one should, of course, supply arguments of the type needed to establish the first similarity rule.

From the second similarity rule one immediately infers the fact that relation (25.29) implies the relation

$$\exp\,[-\mathcal{P}_1]f([\mathcal{P}_2])\exp\,[\mathcal{P}_2] = f[\mathcal{P}_3] \tag{25.31}$$

for every polynomial f. The same relation also holds for functions f for which $f([\mathcal{P}_2])$ and $f([\mathcal{P}_3])$ can be defined. In particular, we are led to the *third similarity rule*: Relation (25.29) implies relation

$$\exp\,[-\mathcal{P}_1]\exp\,[\mathcal{P}_2]\exp\,[\mathcal{P}_1] = \exp\,[\mathcal{P}_3]. \tag{25.32}$$

It is interesting to observe that relation (25.29) implies relation

$$\exp\,(-\mathcal{P}_1)f(\mathcal{P}_2)\exp\mathcal{P}_1 = f(\mathcal{P}_3); \tag{25.33}$$

this relation is immediately verified for polynomials f and therefore holds also for more general functions f. In particular, we have

$$\exp(-\mathcal{P}_1)\exp\mathcal{P}_2\exp\mathcal{P}_1 = \exp\mathcal{P}_3\,. \tag{25.34}$$

In conjunction with this relation, identity (25.32) becomes a special case of the corollary to the composition rule (24.7)′. This special case is thus derived independently of the—unproved—general composition rule.

We shall use the third similarity rule in establishing the identity of the fourth and the second forms of the transformation T.

Composition Rule for Biquantized Operators

We shall need another special case of the composition rule, which also can be proved without the general rule, namely the case in which the pseudo-antisymmetric pseudo-operators $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_{12}$ have only diagonal terms; i.e., they are of the form

$$\mathcal{F} = \begin{pmatrix} F & 0 \\ 0 & -'F \end{pmatrix}. \tag{25.35}$$

The rule then states that relation

$$\exp\mathcal{F}_1\exp\mathcal{F}_2 = \exp\mathcal{F}_{12} \tag{25.36}$$

for three such pseudo-operators entails relation

$$\exp[\mathcal{F}_1]\exp[\mathcal{F}_2] = \exp[\mathcal{F}_{12}]. \tag{25.37}$$

Clearly, relation (25.36) consists of the pair of relations

$$\exp F_1\exp F_2 = \exp F_{12} \tag{25.38}$$

and

$$\exp(-'F_1)\exp(-'F_2) = \exp(-'F_{12}); \tag{25.38$'$}$$

the second is evidently a consequence of the first. By (23.22), (23.27) the composition formula (25.37) can be written as

$$\exp\{A^+F_1A^- \mp \tfrac{1}{2}\,Tr\,F_1\}\exp\{A^+F_2A^- \mp \tfrac{1}{2}\,Tr\,F_2\} = \exp\{A^+F_{12}A^- \mp \tfrac{1}{2}\,Tr\,F_{12}\};$$

because of the composition rule for traces (24.41), (24.42), it is equivalent with the composition rule

$$\exp\{A^+F_1A^-\}\exp\{A^+F_2A^-\} = \exp\{A^+F_{12}A^-\} \tag{25.39}$$

for ordinary biquantized operators. It is, therefore, sufficient to derive formula (25.39) from (25.38).

For this purpose we shall use the particle representation of the states Φ,

$$\Phi \leftrightarrow \{\psi_n(s)_n\}.$$

The representation

$$A^+FA^-\Phi \leftrightarrow \{(F^{s_1} + \cdots + F^{s_n})\psi_n(s)_n\}$$

of the state $A^+FA^-\Phi$, cf. (6.35), involves the operators F^{s_ν} which act on the function $\psi_n(s)_n$ only in so far as this function depends on s_ν. In the representation of $\exp\{A^+FA^-\}\ \Phi$,

$$\exp\{A^+FA^-\}\Phi \leftrightarrow \{\exp(F^{s_1} + \cdots + F^{s_n})\psi_n(s)_n\}, \tag{25.40}$$

cf. (6.36), we may write

$$\exp(F^{s_1} + \cdots + F^{s_n}) = \exp(F^{s_1}) \cdots \exp(F^{s_n}). \tag{25.41}$$

since the operators $F^{s_1}, \cdots, F^{s_n}$ commute.

Moreover, the operators $F_1^{s_\alpha}$ and $F_2^{s_\beta}$, formed from two operators F_1 and F_2, commute even if F_1 and F_2 do not, provided $\alpha \neq \beta$; for, the two operators then act on functions of different variables. Consequently, we have the identity

$$\begin{aligned}\exp(F_1^{s_1} + \cdots + F_1^{s_n}) \exp(F_2^{s_1} + \cdots + F_2^{s_n}) \\ &= \exp F_1^{s_1} \cdots \exp F_1^{s_n} \exp F_2^{s_1} \cdots \exp F_2^{s_n} \\ &= \exp F_1^{s_1} \exp F_2^{s_1} \cdots \exp F_1^{s_n} \exp F_2^{s_n}.\end{aligned}$$

Applying relation (25.38) and relation (25.41) to $F = F_{12}$ we obtain

$$\exp(F_1^{s_1} + \cdots + F_1^{s_n}) \exp(F_2^{s_1} + \cdots + F_2^{s_n}) = \exp(F_{12}^{s_1} + \cdots + F_{12}^{s_n}). \tag{25.42}$$

Since, according to (25.40), the states $\exp\{A^+F_1A^-\}\Phi$, $\exp\{A^+F_2A^-\}\Phi$ and $\exp\{A^+F_{12}A^-\}\Phi$ have the representations

$$\begin{aligned}\exp\{A^+F_1A^-\} \exp\{A^+F_2A^-\}\Phi \\ &\leftrightarrow \{\exp(F_1^{s_1} + \cdots + F_1^{s_n}) \exp(F_2^{s_1} + \cdots + F_2^{s_n})\psi_n(s)_n\}\end{aligned}$$

and

$$\exp\{A^+F_{12}A^-\}\Phi \leftrightarrow \{\exp(F_{12}^{s_1} + \cdots + F_{12}^{s_n})\psi_n(s)_n\},$$

identity (25.42) insures that these two states are the same. Thus the composition rule (25.39) is proved and rule (25.37) follows immediately.

It is remarkable that the derivation of the composition rule for biquantized operators from a representation makes its validity rather obvious while it does not seem obvious how to derive this rule by essentially algebraic means legalizing the use of Baker and Hausdorff's theorem.

Identity of the Fourth and Second Forms of the Operator T

In order to show that the fourth form

$$T_4 = \exp[{}_0\mathcal{F}] \exp[{}_1\mathcal{E}] \exp[{}_1\mathcal{F}] \exp[{}_1\mathcal{G}]$$

of the operator T, cf. (25.6), is identical with the second form

$$T_2 = \exp[\mathcal{E}] \exp[\mathcal{F}] \exp[\mathcal{G}],$$

cf. (24.16), we first verify the identity

$$\exp {}_0\mathcal{F} \exp {}_1\mathcal{E} = \exp \mathcal{E} \exp {}_0\mathcal{F}. \tag{25.43}$$

This is immediately done by writing the pseudo-operator $\exp {}_0\mathcal{F}$ in the form

$$\exp {}_0\mathcal{F} = \begin{pmatrix} Y_{--}[1 - G\bar{G}]^{1/2} & 0 \\ 0 & Y_{++}[1 - \bar{G}G]^{1/2} \end{pmatrix}, \tag{25.44}$$

cf. (25.9), $(25.8)_0$, (25.11) and the pseudo-operators $\exp \mathcal{E}$, $\exp {}_1\mathcal{E}$ in the forms

$$\exp \mathcal{E} = \begin{pmatrix} 1 & 0 \\ Y_{+-}Y_{--}^{-1} & 1 \end{pmatrix},$$

cf. (24.27), $(24.31)_2$,

$$\exp {}_1\mathcal{E} = \begin{pmatrix} 1 & 0 \\ \bar{G} & 1 \end{pmatrix},$$

cf. (25.20), using identity $(25.15)_1$ and setting $\bar{G} = Y_{++}^{-1} Y_{+-}$ by (24.33).

The third similarity rule (25.32) is then applicable to $\mathcal{P}_1 = -{}_0\mathcal{F}$, $\mathcal{P}_2 = {}_1\mathcal{E}$ and $\mathcal{P}_3 = \mathcal{E}$; combined with (25.6) it leads to the form

$$T_4 = \exp [\mathcal{E}] \exp [{}_0\mathcal{F}] \exp [{}_1\mathcal{F}] \exp [{}_1\mathcal{G}].$$

We next employ the decomposition formula

$$\exp {}_0\mathcal{F} \exp {}_1\mathcal{F} = \exp \mathcal{F}, \tag{25.45}$$

which follows from (25.44), (25.21) and

$$\exp \mathcal{F} = \begin{pmatrix} Y_{--} & 0 \\ 0 & 'Y_{--}^{-1} \end{pmatrix},$$

cf. (24.28), (24.30), (25.11). Now we can apply the composition rule (25.37) for biquantized operators and obtain the expression

$$T_4 = \exp [\mathcal{E}] \exp [\mathcal{F}] \exp [{}_1\mathcal{G}]$$

for the fourth form of T, which is the same as the second form because ${}_1\mathcal{G} = \mathcal{G}$, see (25.17). Thus we have shown that the fourth form of T is identical with the second without using the general composition rule. Since the fourth form is identical with the third form which is evidently unitary, we conclude that the second form of T is unitary.

26. *Application to Boson Fields*

Having found the desired canonical transformation in general we proceed to discuss its specific nature in the original problem of a boson field with spring forces.

Reduction of the Quantized to the Unquantized Field Problem

The pseudo-operator $\mathcal{Y}$ associated with the boson field with spring forces is given by

$$(26.1)\qquad \begin{aligned} Y_{--} &= Y_{++} = \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} + \Upsilon^{-1/2}\Omega^{1/2}] \\ Y_{+-} &= Y_{-+} = \tfrac{1}{2}[\Upsilon^{1/2}\Omega^{-1/2} - \Upsilon^{-1/2}\Omega^{1/2}], \end{aligned}$$

cf. (22.22). Here Υ and Ω are the modified and unmodified energy operators for single particles, connected by the relation

$$(26.2)\qquad \Upsilon^2 = \Omega^2 + Q$$

in which the operator Q corresponds to the "spring constant;" cf. (22.15).

It is therefore possible to express the operator $G = Y_{--}^{-1}\, Y_{-+}$ and various other operators that played a role in earlier discussions in terms of Ω and Υ. For this purpose it is convenient to introduce the operators

$$(26.3)\qquad \Gamma = \Omega^{-1/2}\Upsilon\Omega^{-1/2}$$

and

$$(26.4)\qquad Z = \Upsilon^{1/2}\Omega^{-1/2}\Gamma^{-1/2}.$$

Clearly, the operator Γ is Hermitian while Z is unitary; for,

$$Z^* = \Gamma^{-1/2}\Omega^{-1/2}\Upsilon^{1/2},$$

hence

$$Z^*Z = \Gamma^{-1/2}\Gamma\Gamma^{-1/2} = 1$$

and

$$ZZ^* = \Upsilon^{1/2}\Omega^{-1/2}(\Omega^{1/2}\Upsilon^{-1}\Omega^{1/2})\Omega^{-1/2}\Upsilon^{1/2} = 1.$$

The operators Y_{--} and Y_{-+} may be written in terms of Γ and Z as

$$(26.5)\qquad Y_{--} = \tfrac{1}{2}Z\Gamma^{-1/2}[\Gamma + 1], \qquad Y_{-+} = \tfrac{1}{2}Z\Gamma^{-1/2}[\Gamma - 1],$$

the operator G, defined by (24.31), may be written simply as

$$(26.6)\qquad G = [\Gamma + 1]^{-1}[\Gamma - 1]$$

and, since $\overline{G} = G$, the operator $1 - G\overline{G}$ becomes

$$(26.7)\qquad 1 - G\overline{G} = 4\Gamma[\Gamma + 1]^{-2}.$$

Finally we note that the unitary operator $\exp {}_0F$, defined by (24.40), is exactly the operator Z

$$(26.8)\qquad Z = \exp {}_0F.$$

Suppose the operators Ω and Υ are "known," so that the operators Z, Γ, and hence ${}_0F$, $G = \overline{G}$, and so on, can be formed. Assuming that these operators satisfy the conditions formulated earlier, the canonical transformation T in any of its various forms can be set up. By expressing T in terms of Ω and Υ, we may also say that *the problem of modifying the quantized field has been reduced to the problem of modifying the unquantized field.*

Conditions for the Existence of the Canonical Transformation

Necessary and sufficient conditions to be imposed on the operators G and Z were discussed at the end of Section 24. These conditions are equivalent with restrictions on the disturbance operator Q; it does not seem to be easy, however, to give to these restrictions on Q a simple, directly verifiable, form. Nevertheless, certain general statements can be made about them; in doing this it is convenient to work with the operator Γ instead of G.

The conditions $\| G\overline{G} \| < 1$ and $Tr\, G\overline{G} < \infty$, which were introduced at the end of Section 24, cf. (24.47), (24.48), are essentially equivalent with the condition

$$Tr\,(\Gamma - 1)^2 < \infty, \tag{26.9}$$

as could be derived from the formula

$$\Gamma - 1 = 2G[1 - G]^{-1}. \tag{26.10}$$

The operators Υ and Q are easily expressed in terms of the operators Γ and Ω by

$$\Upsilon = \Omega^{1/2}\Gamma\Omega^{1/2} = \Omega^{1/2}(\Gamma - 1)\Omega^{1/2} + \Omega \tag{26.11}$$

and

$$\begin{aligned} Q &= \Omega^{1/2}\Gamma\Omega\Gamma\Omega^{1/2} - \Omega^2 \\ &= \Omega^{1/2}(\Gamma - 1)\Omega(\Gamma - 1)\Omega^{1/2} + \Omega^{3/2}(\Gamma - 1)\Omega^{1/2} + \Omega^{1/2}(\Gamma - 1)\Omega^{3/2}. \end{aligned} \tag{26.12}$$

Special Cases

The requirements on the operator Γ formulated above exclude certain types of disturbance or—in the case of several media—interaction. For example, *homogeneous interaction is excluded.*

We say an operator Λ is "homogeneous" if it commutes with the momentum; for example, Λ is homogeneous if its x-representer is an integral operator with a kernel $\Lambda(x', x'')$ which depends only on the difference $x' - x''$, i.e. if

$$\Lambda(x', x'') = \Lambda(x' - x''). \tag{26.13}$$

Here it is implied that the field extends over the infinite space.

If the interaction operator Q is homogenous then Γ is also homogeneous; for, Q commutes with Ω in this case and the operator Γ is therefore given by $\Gamma = 1 + \Omega^{-2}Q$ as seen from (26.12). Evidently, the kernel $\gamma(x', x'') = \gamma(x' - x'')$ of the operator Γ does not satisfy the condition $Tr\ (\Gamma - 1)^2 < \infty$ since $\iint \gamma^2\,(x' - x'')\,dx'\,dx'' = \infty$.

Another noteworthy fact is that *one-dimensional media with vanishing rest mass μ are excluded if the spring constants are non-negative everywhere.*

To verify this statement we may introduce the Fourier transforms $\tilde{q}(k', k'')$ and $\tilde{\gamma}(k', k'')$ of the kernels $q(x', x'')$ and $\gamma(x', x'')$ of the operators Q and $\Gamma - 1$. Since (26.12) holds and $\mu = 0$ these kernels are related by

$$\tilde{q}(k', k'') = | k'k'' |^{1/2} (| k' | + | k'' |)\tilde{\gamma}_2(k', k'') + | k'k'' |^{1/2} \int \tilde{\gamma}(k', k) \, | k | \, \tilde{\gamma}(k, k'') \, dk.$$

Since the x- and k-spaces are assumed one-dimensional, condition

$$\iint | \tilde{\gamma}(k', k'') |^2 \, dk' \, dk'' < \infty$$

implies that $| k'k'' |^{1/2} \tilde{\gamma}(k', k'')$ vanishes as $| k' |$ and $| k'' | \to 0$. Consequently, $\tilde{q}(0, 0) = 0$. This relation now is equivalent with

$$\iint q(x', x'') \, dx' \, dx'' = 0$$

in contradiction to the assumption that $q(x', x'')$ is non-negative.

Since the failure is due to the behaviour of the function $\tilde{q}(k', k'')$ for $k' = k'' = 0$ we may say that a non-negative spring constant causes an infrared catastrophe in a one dimensional medium. One of the infinities occurring in Sollfrey's problem is due, essentially, to this fact.

Finally we remark that two- or more dimensional media are excluded if the springs which provide the interaction are attached to isolated points. Ultraviolet catastrophes would occur in such cases.

We now consider an example which is very unrealistic physically but can be handled explicitly. In this example the x-representer of the operator $\Upsilon - \Omega$ if of the form

$$\zeta(x', x'') = \iota\zeta(x')\zeta(x''), \tag{26.14}$$

in which the real valued function $\zeta(x)$ and the constant ι are at our disposal. The case in which the interaction operator Q is itself of such a form could also be treated explicitly but would be more complicated and equally unrealistic.

Instead of the x-representers we prefer to work with the energy representers, setting

$$\hat{\zeta}(\omega', \omega'') = \iota\hat{\zeta}(\omega')\hat{\zeta}(\omega'') \tag{26.15}$$

instead of (26.14). We assume that the energy is connected with the position x in a real manner. The energy range is $\mu < \omega < \infty$ where μ is the rest mass, which may be zero or positive. The condition that the operator with the kernel (26.15) produces quadratically integrable functions is

$$\int_{\mu}^{\infty} | \hat{\zeta}(\omega) |^2 \, d\omega < \infty; \tag{26.16}$$

this condition must therefore be imposed. We must also require that the operator Υ be positive definite. This is automatically the case if $\iota > 0$, but if $\iota < 0$ we need the relation

$$-\iota \int_{\mu}^{\infty} \omega^{-1} \mid \hat{\zeta}(\omega) \mid^2 d\omega < 1, \qquad \iota < 0. \tag{26.17}$$

From relation (26.11) we see that the kernel of the operator Γ^{Ω} is

$$\Gamma(\omega', \omega'') = \iota(\omega'\omega'')^{-1/2}\hat{\zeta}(\omega')\hat{\zeta}(\omega''),$$

provided this operator exists; this is the case if

$$\int_{\mu}^{\infty} \omega^{-1} \mid \hat{\zeta}(\omega) \mid^2 d\omega < \infty. \tag{26.18}$$

Since the trace of the square of the operator Γ is exactly the square of the left member of (26.17), this operator is square traceable if it exists. By virtue of (26.17) this condition is automatically satisfied if $\iota < 0$ or if $\mu > 0$. If $\iota > 0$ and $\mu = 0$, however, functions $\hat{\zeta}(\omega)$ can be defined which satisfy condition (26.16) but not condition (26.18). Thus we see *that it may happen that the operator* Υ *is defined and has a spectrum of the same type as* Ω *and still the canonical transformation* T *does not exist.*

If $\iota < 0$ and $\mu > 0$ the operator Υ may possess a point eigenvalue v_0. The condition for the occurrence of such a point eigenvalue is evidently the existence of a quadratically integrable function $\psi_0(\omega)$ for which

$$(\omega - v_0)\psi_0(\omega) + \iota\hat{\zeta}(\omega) \int_{\mu}^{\infty} \hat{\zeta}(\omega)\psi_0(\omega)\, d\omega = 0.$$

This condition is satisfied for

$$\psi_0(\omega) = c(\omega - v_0)^{-1}\hat{\zeta}(\omega) \tag{26.19}$$

provided the value $v = v_0$ can be so chosen in the interval $0 < v < \mu$ that

$$-\iota \int_{\mu}^{\infty} (\omega - v)^{-1} \mid \hat{\zeta}(\omega) \mid^2 d\omega = 1. \tag{26.20}$$

When v varies from 0 to μ the left member of this equation varies from the left member of condition (26.17) to the left member of condition

$$-\iota \int_{\mu}^{\infty} (\omega - \mu)^{-1} \mid \hat{\zeta}(\omega) \mid^2 d\omega > 1, \qquad \iota < 0. \tag{26.18}$$

Clearly, if this latter condition is satisfied a point eigenvalue $v = v_0$ exists.

Thus we see that *it may happen that the spectra of the operators* Υ *and* Ω *are of different types and still the canonical transformation* T *does exist.*

The existence of a point eigenstate or "bound" state Ψ_0 of the operator Υ entails the existence of bound states of the modified field. If $\psi_0(x)$ is the representer of Ψ_0 and v_0 the eigenvalue, the state Φ_n of the field with the particle representer $\psi_n(s)_n = \psi_0(s_1) \cdots \psi_0(s_n)$ is such a bound state with the eigenvalue $n\,v_0$. One may say that in this state the field consists of n particles in the state Ψ_0. Thus we see that *the canonical transformation may exist even in cases in which the modified field possesses bound states while the unmodified field does not.*

27. *Transition Operator. Scattering Operator*

In this section we shall calculate the transition amplitudes connecting the state of the field at two different times. We can do this in two ways: Either we can employ the "method of spectral transformation" making use of the fact that variation in time of the modified particle representers can immediately be determined by applying the operator $\exp\{-itH_B\}$. Or we may utilize the fact that the annihilation operators at two different times are connected by a transformation similar to that which connects modified and unmodified operators of this kind.

We shall also study the asymptotic behavior of the transition amplitudes and, in particular, give a simple asymptotic description of the incomplete scattering operator in terms of "original but scattered," "spontaneously emitted and half scattered," and "spontaneously emitted but not scattered" particles. We shall see that no complete scattering operator in the strict sense exists, but that nevertheless the transition probabilities approach definite limits as the time interval increases indefinitely.

Method of Spectral Transformation

We employ the canonical transformation T in order to express the particle representers $\psi_n(s)_n(t)$ at any time t in terms of the "$N(0)$-representers" $\psi_n(s)_n(0)$, see (22.27). We consider the functions $\psi_n(s)_n(t)$ as the $N(0)$-representers of a time-dependent Schrödinger state $\Phi_s(t)$, cf. Section 15 footnote 4 and (15.15). As seen from (22.28), (22.32), (22.17), this state is given by

$$\Phi_s(t) = T^{-1} \exp\{-itA^+ \Upsilon A^-\} T\Phi. \tag{27.1}$$

Since the relation

$$\exp\{-itA^+ \Upsilon A^-\} T = T \exp\{-itB^+ \Upsilon B^-\} \tag{27.2}$$

can be derived from the relation $A^{\pm}T = TB^{\pm}$, see (22.34), the state $\Phi_s(t)$ can also be expressed as

$$\Phi_s(t) = \exp\{-itB^+ \Upsilon B^-\}\Phi = \exp\{-itH_B\}\Phi, \tag{27.3}$$

and thus be recognized as the solution of the Schrödinger equation

$$i\nabla_t \Phi_s(t) = H_B \Phi_s(t). \tag{27.4}$$

Instead of using $\Phi_s(t)$ we prefer to use the corresponding interaction state $\Phi_w(t)$ given by

$$\Phi_w(t) = W(t)\Phi \tag{27.5}$$

when we set

$$W(t) = \exp\{itA^{+}\Omega A^{-}\}T^{-1}\exp\{-itA^{+}\Upsilon A^{-}\}T. \tag{27.6}$$

or

$$W(t) = \exp\{itA^{+}\Omega A^{-}\}\exp\{-itB^{+}\Upsilon B^{-}\}. \tag{27.7}$$

Transition Operator

Operators of the type $\exp A^{+}\Lambda A^{-}$ will occur occasionally in the following. Since we want to use our similarity and composition rules for pseudo-biquantized operators we introduce to every operator Λ the pseudo-antisymmetric pseudo-operator

$$\hat{\Lambda} = \begin{pmatrix} \Lambda & 0 \\ 0 & -{}'\Lambda \end{pmatrix} \tag{27.8}$$

and use the relation

$$\exp[\hat{\Lambda}] = \exp\{\tfrac{1}{2}A\hat{\Lambda}A\} = \exp\{\tfrac{1}{2}\,Tr\,\Lambda\}\exp A^{+}\Lambda A^{-}. \tag{27.9}$$

In particular, although the traces $Tr\,\Omega$ and $Tr\,\Upsilon$ do not exist, we shall introduce the pseudo-operators $\hat{\Omega}$ and $\hat{\Upsilon}$ and furthermore the field operator

$$\tilde{T}^{-1}(t) = \exp\{it[\hat{\Omega}]\}T^{-1}\exp\{-it[\hat{\Upsilon}]\}T, \tag{27.10}$$

which we shall call the "transition operator." We have reserved the notation $\tilde{T}(t)$ for its inverse, the operator

$$\begin{aligned} \tilde{T}(t) &= T^{-1}\exp\{it[\hat{\Upsilon}]\}T\exp\{-it[\hat{\Omega}]\} \\ &= \exp\{it[\hat{\Omega}]\}\tilde{T}^{-1}(-t)\exp\{-it[\hat{\Omega}]\} \end{aligned} \tag{27.10$_1$}$$

in order to facilitate the description of the decomposition formulas.

If the operator $\tilde{T}^{-1}(t)$ existed it would differ from the operator $W(t)$ by the factor $\exp\{-ith\}$, as seen from (27.9) for $\Lambda = \Omega$ and $\Lambda = \Upsilon$; i.e.,

$$\tilde{T}^{-1}(t) = \exp\{-ith\}W(t), \tag{27.11}$$

with $h = \frac{1}{2}\,Tr\,(\Upsilon - \Omega)$, cf. (22.26). Assuming that this difference trace exists, we may define $\tilde{T}(t)$ by $(27.10)_1$ and employ our rules as we would if the expression $(27.10)_1$ were defined. The resulting identities must, however, be verified by some other method. The state

$$\tilde{\Phi}_w(t) = \exp\{-ith\}\Phi_w(t) = \tilde{T}^{-1}(t)\Phi \tag{27.12}$$

will be called the "adjusted interaction state."

Another form of the transition operator is

$$\tilde{T}^{-1}(t) = \exp\left\{\frac{it}{2}\,\mathfrak{A}\circ\hat{\Omega}\mathfrak{A}\right\}\exp\left\{-\frac{it}{2}\,\mathfrak{B}\circ\hat{\Upsilon}\mathfrak{B}\right\}. \tag{27.13}$$

this formula is identical with (27.10), as we may deduce from the relation

$$['\mathcal{Y}\hat{\mathcal{T}}\mathcal{Y}] = \tfrac{1}{2}\mathcal{B} \circ \hat{\mathcal{T}}\mathcal{B} \tag{27.14}$$

or

$$\mathcal{A} \circ '\mathcal{Y}\hat{\mathcal{T}}\mathcal{Y}\mathcal{A} = \mathcal{Y}\mathcal{A} \circ \hat{\mathcal{T}}\mathcal{Y}\mathcal{A} = \mathcal{B} \circ \hat{\mathcal{T}}\mathcal{B},$$

which follows from (22.21). We need only use the first form $T = T_1 = \exp\{[\log \mathcal{Y}]\}$ of T and the third similarity rule (25.32), (25.29), with $\mathcal{P}_1 = \log \mathcal{Y}$, $\mathcal{P}_2 = \hat{\mathcal{T}}$.

Direct Method

Expressions for the operator $\tilde{T}(t)$ can also be derived by recognizing that the annihilation and creation operators $A^{\mp}(s, t)$ at the time t and those at the time $t = 0$ are connected by a homogeneous linear relation[8] of the same type as those which connect the operators $B^{\mp}$ with the operators $A^{\mp}$.

We note that relation (22.19), with s instead of x, can be written in the form

$$\mathcal{B}(s, t) = \exp\{-it\hat{\mathcal{T}}'\}\mathcal{B}(s, 0)$$

or simply as

$$\mathcal{B}(t) = \exp\{-it\hat{\mathcal{T}}\}\mathcal{B}(0). \tag{27.15}$$

If we insert this expression into (22.24) and then insert the result into (22.21) for $t = 0$ we obtain, by (23.20)′,

$$\mathcal{A}(t) = '\mathcal{Y} \exp\{-it\hat{\mathcal{T}}\}\mathcal{Y}\mathcal{A}(0). \tag{27.16}$$

Since we would like to employ our rules, we shall work with the "adjusted" annihilation and creation operators $\tilde{A}^{\mp}(s, t)$, given by

$$\tilde{\mathcal{A}}(t) = \exp\{it\hat{\Omega}\}\mathcal{A}(t) \tag{27.17}$$

or

$$\tilde{\mathcal{A}}(t) = '\tilde{\mathcal{Y}}(t)\mathcal{A}(0) \tag{27.18}$$

with

$$'\tilde{\mathcal{Y}}(t) = \exp\{it\hat{\Omega}\}'\mathcal{Y} \exp\{-it\mathcal{T}\}\mathcal{Y}, \tag{27.19}$$

or

$$\begin{aligned} \tilde{\mathcal{Y}}(t) &= '\mathcal{Y} \exp\{it\hat{\mathcal{T}}\}\mathcal{Y} \exp\{-it\hat{\Omega}\} \\ &= \exp\{it\hat{\Omega}\}'\tilde{\mathcal{Y}}(-t) \exp\{-it\hat{\Omega}\}. \end{aligned} \tag{27.19$_1$}$$

---

[8] This is true even if the disturbing operator Q is not independent of the time. We shall not treat this case here, although it would be interesting to do so, in particular, since the results could presumably be cast into a Lorentz invariant form. This case will be treated in a forthcoming paper by B. Zumino.

The theory developed in the preceding sections can be applied to this transformation $\tilde{\mathcal{Y}}(t)$ since it evidently satisfies the condition $\tilde{\mathcal{Y}}(t)'\tilde{\mathcal{Y}}(t) = 1$. Without assuming that the operator $\mathcal{Y}(t)$ is of the form $\exp \tilde{\mathcal{R}}(t)$, we may find the corresponding canonical transformation by the composition rule. On the right member of $(27.19)_1$ we replace $'\mathcal{Y}$, $\exp \{it\hat{\Upsilon}\}$, $\mathcal{Y}$, and $\exp \{-it\hat{\Omega}\}$ by $\exp [\log {'\mathcal{Y}}] = T^{-1}$, $\exp \{it[\hat{\Upsilon}]\}$, $\exp [\log \mathcal{Y}] = T$, and $\exp \{-it[\hat{\Omega}]\}$ respectively. The resulting canonical transformation is seen to be exactly the field operator $\tilde{T}(t)$, given by $27.10)_1$. Relation (27.18) leads therefore to relation

$$\tilde{\mathcal{A}}(t) = \tilde{T}(t)\mathcal{A}(0)\tilde{T}^{-1}(t) \tag{27.20}$$

from which we deduce that *the* $\mathcal{A}(0)$*-representation of the adjusted interaction state* Φ_w, cf. (27.12), *gives a* $\tilde{\mathcal{A}}(t)$*-representation of the state* Φ.

Properties of the Transition Operator

We proceed to calculate various quantities which will enable us to analyze the nature of the inverse transition operator $\tilde{T}(t)$. We shall in particular refer to the second form

$$\tilde{T}(t) = \exp [\tilde{\mathcal{E}}(t)] \exp [\tilde{\mathcal{F}}(t)] \exp [\tilde{\mathcal{G}}(t)] \tag{27.21}$$

of this operator, see (24.18). From (27.19) we find

$$\begin{aligned} \tilde{Y}_{--}(t) &= ['Y_{++} \exp \{it\Upsilon\} Y_{--} - 'Y_{-+} \exp \{-it\Upsilon\} Y_{+-}] \exp \{-it\Omega\} \\ \tilde{Y}_{-+}(t) &= ['Y_{++} \exp \{it\Upsilon\} Y_{-+} - 'Y_{-+} \exp \{-it\Upsilon\} Y_{++}] \exp \{it\Omega\} \end{aligned} \tag{27.22}$$

or, from (24.35), (24.34),

$$\begin{aligned} \tilde{Y}_{--}(t) &= 'Y_{++}[\exp \{it\Upsilon\} - \bar{E} \exp \{-it\Upsilon\}E] Y_{--} \exp \{-it\Omega\}, \\ \tilde{Y}_{-+}(t) &= 'Y_{++}[\exp \{it\Upsilon\}\bar{E} - \bar{E} \exp \{-it\Upsilon\}] Y_{++} \exp \{it\Omega\}, \end{aligned}$$

and, finally,

$$\begin{aligned} \tilde{Y}_{--}(t) &= 'Y_{++} \exp \{it\Upsilon\}[1 - \bar{E}_\Upsilon(t)E] Y_{--} \exp \{-it\Omega\}, \\ \tilde{Y}_{-+}(t) &= 'Y_{++} \exp \{it\Upsilon\}[\bar{E} - \bar{E}_\Upsilon(t)] Y_{++} \exp \{it\Omega\} \end{aligned} \tag{27.23}$$

when we set

$$E_\Upsilon(t) = \exp \{-it\Upsilon\}\bar{E} \exp \{-it\Upsilon\}. \tag{27.24}$$

For the operator $\tilde{G}(t)$, which determines the creation factor in (27.21), we find by (24.32) the expression $\tilde{G}(t) = \tilde{Y}_{--}^{-1}(t)\tilde{Y}_{-+}(t)$ or

$$\tilde{G}(t) = \exp \{it\Omega\} Y_{--}^{-1}[1 - \bar{E}_\Upsilon(t)E]^{-1}[\bar{E} - \bar{E}_\Upsilon(t)] Y_{++} \exp \{it\Omega\}. \tag{27.25}$$

For the analysis of the time variation of the vacuum state we also need the number $\tilde{\eta}(t) = \exp \{\mp\frac{1}{2}\, Tr\, \tilde{F}(t)\}$, or rather the number

$$| \tilde{\eta}(t) | = \exp \{\mp\tfrac{1}{2}\, Tr \log {'\tilde{Y}_{++}(t)}\tilde{Y}_{--}(t)\}, \tag{27.26}$$

cf. (24.45). We first note the relation

$$'\tilde{Y}_{++}(t)\tilde{Y}_{--}(t) = \exp\{it\Omega\}\tilde{Y}_{++}[1 - \bar{E}E_{\Upsilon}(t)] \exp\{-it\Upsilon\} Y_{--}{}'Y_{++} \exp\{it\Upsilon\} \cdot [1 - \bar{E}_{\Upsilon}(t)E] Y_{--} \exp\{-it\Omega\}$$

and make use of the trace composition rule (24.42). We then find

$$\exp Tr \log {}'\tilde{Y}_{++}(t)\tilde{Y}_{--}(t) = \exp 2\, Tr \log {}'Y_{++}Y_{--} \exp 2\, \mathcal{R}e\, Tr \log[1 - \bar{E}_{\Upsilon}(t)E].$$

Introducing the number $|\eta|$ given by (24.45) we finally obtain

$$|\tilde{\eta}(t)| = |\eta|^2 \exp\{\mp\tfrac{1}{2}\, \mathcal{R}e\, Tr \log[1 - \bar{E}_{\Upsilon}(t)E]\}. \tag{27.27}$$

Since $1 - \bar{E}E = (Y_{--}{}'Y_{++})^{-1}$, cf. (24.39), we have

$$Tr \log {}'Y_{++}Y_{--} = Tr \log Y_{--}{}'Y_{++} = -Tr \log[1 - \bar{E}E]$$

and hence, again by virtue of the trace composition rule, we can write

$$|\eta(t)| = \exp\{\mp\tfrac{1}{2}\, \mathcal{R}e\, Tr \log[1 - \bar{E}E]^{-1}[1 - \bar{E}_{\Upsilon}(t)E]\}. \tag{27.28}$$

Since $\bar{E}_{\Upsilon}(0) = \bar{E}$, it is clear from this expression that

$$|\eta(0)| = 1. \tag{27.29}$$

The question arises: are there any cases in which *the canonical transformation T* (connecting the modified and unmodified representations) *fails to exist although the transition operator $\tilde{T}(t)$* (connecting unmodified representations at different times) *does exist.* We recall that a similar situation could arise in the case of the infrared catastrophe treated in Part IV. In that case the field was myriotic with respect to the modified annihilation and creation operators $B^{\vee}$ while it remained ordinary with respect to the unmodified operators $A^{\vee}$ at all times.

A closer analysis seems to indicate that *such a situation cannot arise in the present case.* For, it seems that the trace of $\mathcal{R}e\,[1 - \bar{E}E]^{-1}[\bar{E}E - \bar{E}_{\Upsilon}(t)E]$ cannot exist if the trace of $\bar{E}E$ does not exist, although the trace of $\mathcal{R}e\,[\bar{E}E - \bar{E}_{\Upsilon}(t)E]$ might exist. Thus it is indicated that *a field which is myriotic with respect to the modified operators $B^{\vee}$ becomes myriotic with respect to $A^{\vee}$ instantaneously for $t > 0$ even if it was amyriotic with respect to $A^{\vee}$ at the time $t = 0$.*

Time Variation of the Vacuum State

Suppose the state Φ is an unmodified vacuum state $\Phi = \Phi_{\text{vac}}$ at the time $t = 0$; then the nature of this state at a time $t > 0$ can be read off from the A-representation of the adjusted interaction state defined by (27.12),

$$\tilde{\Phi}_w(t) = \tilde{\eta}(t) \exp[\tilde{\mathfrak{g}}(t)]\Phi_{\text{vac}}. \tag{27.30}$$

For, as remarked in connection with relation (27.70), the $A(0)$-representation of $\hat{\Phi}(t)$ is the $\bar{A}(t)$-representation of the state Φ. Hence, according to formula (24.26), the number $|\tilde{\eta}(t)|^2$ is *the probability that the field will be in the vacuum state* *at the time t if it was in the vacuum state at the time* $t = 0$. From (27.27) we find the expression

$$P(N(t), 0 \mid N(0), 0) = |\eta|^4 \exp \{\mp \Re e\, Tr \log [1 - \bar{E}_\Upsilon(t)E]^{-1}\}. \tag{27.31}$$

for this probability.

Asymptotic Transition Probabilities

It is not difficult to calculate the limit for $t \to \infty$ of the probability that the field will be in a vacuum state at the time t if it was in such a state at the time $t = 0$, provided the field extends over the infinite space. By making this assumption we exclude, for example, those cases of several interacting media described in Section 22, in which one medium is concentrated at a single point. We may conclude from this assumption that the operator $\bar{E}_\Upsilon(t)E$ approaches zero as $|t| \to \infty$. For, in a representation in which the modified energy Υ is diagonalized, the kernel of the operator $\bar{E}_\Upsilon(t)E$ is

$$\exp \{itv'\} \int \bar{E}(v', v) \exp \{itv\} E(v, v'')\, dv$$

if $E(v', v'')$ is the kernel of E; here we omit reference to the accessory variables which, together with Υ, form a complete system of observables. Since $E(v', v'')$ is assumed to be quadratically integrable, it is clear that the integral approaches zero as $|t| \to \infty$; i.e.,

$$\bar{E}_\Upsilon(t)E \to 0 \qquad \text{as} \qquad |t| \to \infty. \tag{27.32}$$

We recall that a similar argument was used in Part III, cf. (13.45). It follows that $\log [1 - \bar{E}_\Upsilon(t)\, E]$ approaches zero, whence relation (27.27) gives

$$|\eta(t)| \to |\eta|^2 \tag{27.33}$$

or

$$P(N_\infty, 0 \mid N(0), 0) = |\eta|^4. \tag{27.34$_0$}$$

In other words, *the probability that the state will eventually be a vacuum state if it was a vacuum state originally is the square of the probability that it will be found to be a modified vacuum state,* cf. (24.26).

This result is a special case of the formula

$$P(N_\infty, n \mid N(0), n_0) = \sum_m P(N_\infty, n \mid M, m) P(M, m \mid N(0), n_0) \tag{27.34}$$

which could be interpreted by saying that *the classical rule of composition of probabilities holds even if the number M of modified particles is not measured after the initial measurement of the number N of unmodified particles.* A similar result was obtained in Part III, cf. (15.40). The probability $P(N_\infty, n \mid M, m)$ in formula (27.34), the limit of $P(N(t), n \mid M, m)$ as $t \to \infty$, can simply be expressed as[9]

$$P(N_\infty, n \mid M, m) = P(N, m - n \mid M, 0) \tag{27.35}$$

---

[9] Formula (15.40) in Part III should also have been supplemented by formula (27.34). The value of $P(N, m - n \mid M, 0)$ there would be given by (14.55).

and the probability $P(N, m - n \mid M, n)$ is given by formula $(24.26)_1$. We shall not give a proof of formulas (27.34), (27.35) although this could be done easily.

Scattering Operator

For single particles with the energy Υ one may express the Schrödinger state $\Psi_s(t)$ at the time t in terms of the Schrödinger state $\Psi = \Psi_s(0)$ at the time $t = 0$ by the formula

$$\Psi_s(t) = \exp\{-it\Upsilon\}\Psi. \tag{27.36}$$

Considering the operator Υ as a modification of the undisturbed operator Ω one may introduce the interaction state

$$\Psi_w(t) = U(t)\Psi \tag{27.37}$$

with

$$U(t) = \exp\{it\Omega\}\exp\{-it\Upsilon\}. \tag{27.38}$$

Under appropriate conditions on the disturbance $\Upsilon - \Omega$, see, e.g. [45], the operator $U(t)$ approaches limits $U_{\pm}$ as $t \to \pm\infty$,

$$U(t) \to U_{\pm} \quad \text{as} \quad t \to \pm\infty. \tag{27.39}$$

The operators $U_{\pm}$, which may be called "half scattering operators," satisfy the relation, cf. [44] and [45],

$$\Omega U_{\pm} = U_{\pm}\Upsilon. \tag{27.40}$$

The scattering operator, which we here denote by U_{sc}, may then be defined as

$$U_{sc} = U_{+}U_{-}^{-1}; \tag{27.41}$$

it evidently commutes with the undisturbed energy operator Ω,

$$\Omega U_{sc} = U_{sc}\Omega. \tag{27.42}$$

Of course, the scattering operator may also be defined as the limit of the "incomplete scattering operator"

$$U_{sc}(t, t_0) = U(t)U^{-1}(t_0) = \exp\{it\Omega\}\exp\{Ui(t_0 - t)\Upsilon\}\exp\{-it_0\Omega\}, \tag{27.42}$$

i.e.

$$U_{sc}(t, t_0) \to U_{sc} \quad \text{as} \quad t \to \infty,\ t_0 \to -\infty. \tag{27.39$_{sc}$}$$

The analogue of the operator $U(t)$ for the fields under consideration is the operator $\tilde{T}^{-1}(t)$ as seen from (27.13); the analogue of the operator $U_{sc}(t, t_0) = U(t)U^{-1}(t_0)$ is the operator

$$T_{sc}^{-1}(t, t_0) = \tilde{T}^{-1}(t)\tilde{T}(t_0), \tag{27.44}$$

which we also term "incomplete scattering operator." Neither the operators $\tilde{T}(t)$ nor the operators $T_{sc}^{-1}(t, t_0)$ approach limits as $t \to +\infty$, $t_0 \to -\infty$. For

this reason we must be satisfied with an asymptotic description of these operators for large values of t. This description will be given only for $T^{-1}_{sc}(t, t_0)$.
The formula

$$T_{sc}(t, t_0) = \exp\{it_0[\hat{\Omega}]\}\tilde{T}(t - t_0)\exp\{-it_0[\hat{\Omega}]\}, \tag{27.45}$$

which can be deduced from formulas (27.10), will be useful for this purpose.

Scattering Operator According to Yang and Feldman

There is another possibility of introducing the scattering operator. From formula (27.18) we see that the pseudo-operator $\tilde{\mathfrak{A}}(t)$ can be expressed in terms of the pseudo-operator $\mathfrak{A}(t_0)$ by the relation

$$\tilde{\mathfrak{A}}(t) = \tilde{\mathcal{Y}}(t, t_0)\tilde{\mathfrak{A}}(t_0)$$

in which the linear transformation

$$\tilde{\mathcal{Y}}(t, t_0) = {}'\tilde{\mathcal{Y}}(t)\tilde{\mathcal{Y}}(t_0)$$

again satisfies the condition $\tilde{\mathcal{Y}}(t, t_0)'\tilde{\mathcal{Y}}(t, t_0) = 1$. Relation (27.20) leads to the relation

$$\mathfrak{A}(t) = S^{-1}(t, t_0)\mathfrak{A}(t_0)S(t, t_0)$$

in which

$$S(t, t_0) = \tilde{T}(t_0)\tilde{T}^{-1}(t).$$

This operator is therefore the canonical transformation associated with the linear transformation $\tilde{\mathcal{Y}}(t, t_0)$.[10] If the operator $S(t, t_0)$ approached a limit as $t_0 \to -\infty$, $t \to \infty$, this limit would be the scattering operator in the sense of Yang and Feldman, [5].

From the definition of the operator $S(t, t_0)$ it is clear *that the* $\tilde{\mathfrak{A}}(t_0)$*-representation of the state* $S(t, t_0)\,\Phi$ *is a* $\tilde{\mathfrak{A}}(t)$*-representation of the Heisenberg state* Φ.

The connection between the operator $S(t, t_0)$ and the operator $\tilde{T}^{-1}_{sc}(t, t_0)$ introduced above is evidently given by the relation

$$S(t, t_0) = \tilde{T}(t_0)\tilde{T}^{-1}_{sc}(t, t_0)\tilde{T}^{-1}(t_0),$$

in agreement with the fact that the operator $\tilde{T}^{-1}_{sc}(t, t_0)$ transforms the interaction state $\tilde{T}^{-1}(t_0)\,\Phi$ into the interaction state $\tilde{T}^{-1}(t)\,\Phi$.

This relation has the following consequence. Suppose we express the operator $\tilde{T}^{-1}_{sc}(t, t_0)$ in terms of the operators $A^{*}(0)$ as

$$\tilde{T}^{-1}_{sc}(t, t_0) = f(A^{*}(0)),$$

then, by virtue of (27.20),

$$S(t, t_0) = f(\tilde{A}^{*}(t_0)).$$

---

[10] This connection was pointed out to me by B. Zumino. It will be the starting point for his forthcoming paper.

Therefore, in describing the effect of the operator $\tilde{T}^{-1}_{sc}(t, t_0)$ in terms of the $A(0)$-representers of a state we shall have described the effect of the operator $S(t, t_0)$ in terms of the $\tilde{A}(t)$-representers of a state.

Asymptotic Field Scattering Operator

We proceed to show that for large positive values of t and $-t_0$ the operator $T^{-1}_{sc}(t, t_0)$ admits the description

$$T^{-1}_{sc}(t, t_0) = \exp\{-\tfrac{1}{2}A^+G(-t)A^+\}\exp\{\tfrac{1}{2}A^+G_+(-t_0)A^+\} \cdot \exp\{\tfrac{1}{2}i(t - t_0)h\}\exp\{A^+\log U_{sc}A^-\} \tag{27.46}$$

with

$$G(-t) = \exp\{it\Omega\}G\exp\{it\Omega\} \tag{27.47}$$

and

$$G_+(-t_0) = \exp\{it_0\Omega\}U_+\overline{E}'U_+\exp\{it_0\Omega\}. \tag{27.48}$$

Here U_{sc} is the scattering operator and U_+ the half scattering operator for a single particle, cf. (27.39) (27.41). In deriving formula (27.46) we shall make two assumptions. *Assumption I: the field extends over the whole space. Assumption II: the operators Y_{--} and Y_{++} differ from the identity by a square traceable operator.* The latter assumption is satisfied in the problem of the boson field treated in Sections 22 and 26. We shall first justify formula (24.47) and then give an interpretation of it.

Justification

We employ the second form of the operator $T_{sc}(t, t_0)$ given by (24.16) and write it as

$$T_{sc}(t, t_0) = \exp\{[\mathcal{E}_{sc}(t, t_0)]\}\exp\{[\mathcal{F}_{sc}(t, t_0)]\}\exp\{[\mathcal{G}_{sc}(t, t_0)]\}. \tag{27.49}$$

Using the form (27.45) of $T_{sc}(t, t_0)$ and observing (24.30), (24.31) we obtain

$$\begin{aligned} E_{sc}(t, t_0) &= \exp\{-it_0\Omega\}\tilde{E}(t - t_0)\exp\{-it_0\Omega\}, \\ G_{sc}(t, t_0) &= \exp\{it_0\Omega\}\tilde{G}(t - t_0)\exp\{it_0\Omega\}, \\ \exp F_{sc}(t, t_0) &= \exp\{it_0\Omega\}\tilde{Y}_{--}(t - t_0)\exp\{-it_0\Omega\}. \end{aligned} \tag{27.50}$$

From (27.25) we then have

$$G_{sc}(t, t_0) = \exp\{it\Omega\}Y^{-1}_{--}[1 - \overline{E}_\Upsilon(t - t_0)E]^{-1} \cdot [\overline{E} - \overline{E}_\Upsilon(t - t_0)]Y_{++}\exp\{it\Omega\}; \tag{27.51}$$

from (27.23) we have

$$\exp F_{sc}(t, t_0) = \exp\{it_0\Omega\}'Y_{++}\exp\{i(t - t_0)\Upsilon\} \cdot [1 - \overline{E}_\Upsilon(t - t_0)E]Y_{--}\exp\{-it\Omega\}, \tag{27.52}$$

and $E_{se}(t, t_0)$ is given by a formula similar to (27.50).

As a consequence of Assumption I we have $\bar{E}_{\Upsilon}(t)E \to 0$ as before, cf. (27.32). Arguments similar to those used to derive (27.32) yield the relation

$$\text{(27.53)} \qquad \exp\{[\mathcal{E}]\} \exp\{itA^{+}\Omega A^{-}\}\Phi \to \Phi \qquad \text{as} \qquad |t| \to \infty .$$

More specifically this formula can be derived from the fact that

$$\iint E(s_1, s_2) \exp\{it\Omega^{s_1}\} \exp\{it\Omega^{s_2}\}\psi(s_1, s_2)\, dm(s_1)\, dm(s_2) \to 0 \qquad \text{as} \qquad |t| \to 0$$

for every function $\psi(s_1, s)$ with $\iint |\psi(s_1, s_2)|^2\, dm(s_1)\, dm(s_2) < \infty$. From the form (27.50) of the operator $E_{se}(t)$ one may deduce by similar arguments the relation

$$\text{(27.54)} \qquad \exp\{-[\mathcal{E}_{se}(t, t_0)]\}\Phi \to \Phi \qquad \text{as} \qquad |t| \to \infty .$$

In other words, *the annihilation operators do not contribute asymptotically.*

In order to determine the asymptotic contribution from $\exp\{[F_{se}(t)]\}$ we make use of Assumption II and deduce from it the relations

$$\text{(27.55)} \qquad \begin{aligned} &\exp\{it\Omega\} Y_{--} \exp\{-it\Omega\} \to 1 \\ &\exp\{-it\Omega\}' Y_{++} \exp\{it\Omega\} \to 1 \qquad \text{as} \qquad |t| \to \infty . \end{aligned}$$

From (27.52) we see that the operator $\exp F_{se}(t)$ is given asymptotically by

$$\exp F_{se}(t, t_0) \sim \exp\{it_0\Omega\} \exp\{i(t - t_0)\Upsilon\} \exp\{-it\Omega\}$$

or, because of (24.44), by

$$\text{(27.56)} \qquad \exp F_{se}(t, t_0) \sim U_{se}^{-1}(t, t_0) \qquad \text{as} \qquad t \sim \infty, t \sim -\infty .$$

Consequently, because of (27.09), we have

$$\begin{aligned} \exp\{[F_{se}(t, t_0)]\} &\sim \exp\{-[\log U_{se}(t, t_0)]\} \\ &= \exp\{-\tfrac{1}{2} Tr \log U_{se}(t, t_0)\} \exp\{-A^{+} \log U_{se}(t, t_0) A^{-}\} . \end{aligned}$$

Now,

$$Tr \log U_{se}(t, t_0) = -i(t - t_0)\, Tr\, (\Upsilon - \Omega) = -i(t - t_0)h$$

as seen from (27.43). Hence, because of $(27.39)_{se}$, relation

$$\text{(27.57)} \qquad \exp\{[F_{se}(t, t_0)]\} \sim \exp\{\tfrac{1}{2}i(t - t_0)h\} \exp\{-A^{+} \log U_{se} A^{-}\}$$

follows.

From relation (27.51) we finally have

$$\begin{aligned} G_{se}(t, t_0) \sim{} & \exp\{it\Omega\} Y_{--}^{-1} \bar{E} Y_{++} \exp\{it\Omega\} \\ & - \exp\{it\Omega\} Y_{--}^{-1} \exp\{-i(t - t_0)\Upsilon\} \bar{E} \exp\{-i(t - t_0)\Upsilon\} Y_{++} \exp\{it\Omega\} . \end{aligned}$$

Now, $Y_{--}^{-1}\bar{E}Y_{++} = G$ by (24.32), (24.35); hence, by (27.55), (27.38), (27.39) and (27.47), (27.48)

$$G_{sc}(t, t_0) \sim G(-t) - G_{+}(-t_0). \tag{27.58}$$

In virtue of $\exp(\mathcal{G}_1 + \mathcal{G}_2) = \exp \mathcal{G}_1 \exp \mathcal{G}_2$ and $[\mathcal{G}] = \frac{1}{2}A^{+}GA^{+}$ formula (27.46) follows.

Interpretation

The operator $\exp\{A^{+} \log U_{sc}A^{-}\}$ which occurs in formula (24.47) will, somewhat improperly, be called the *biquantized scattering operator.* For, this operator transforms the state Φ_w, represented as

$$\Phi_w \leftrightarrow \{\psi_n(s)_n\} \tag{27.60}$$

into the state which is represented as

$$\exp\{A^{+} \log U_{sc}A^{-}\}\Phi_w \leftrightarrow \{U_{sc}^{s_1} \cdots U_{sc}^{s_n}\psi_n(s)_n\}; \tag{27.60$_1$}$$

thus *the representers of this state are obtained by letting the operator U_{sc} act on the representers $\psi_n(s_1, \cdots, s_n)$ considered successively as functions of $s_1, \cdots, s_n$.*

In order to describe the nature of the operator $\exp\{\frac{1}{2}A^{+}G(-t)A^{+}\}$, which also occurs in formula (24.42), we first observe that it transforms the vacuum state Φ_{vac} into the state represented as

$$\begin{aligned} &\exp\{\tfrac{1}{2}A^{+}G(-t)A^{+}\}\Phi_{vac} \\ &\qquad \leftrightarrow \left\{(n!)^{1/2}\begin{pmatrix}\text{Sy}\\ \text{Asy}\end{pmatrix} g(s_1, s_2, -t) \cdots g(s_{n-1}, s_n, -t)\right\}. \end{aligned} \tag{27.61}$$

Here

$$g(s_1, s_2, -t) = \exp\{it\Omega^{s_1}\} \exp\{it\Omega^{s_2}\}g(s_1, s_2) \tag{27.62}$$

is the kernel of the integral operator $G^{s}(-t)$. Similarly, the operator $\exp\{\frac{1}{2}A^{+}G_{+}(-t_0)A^{+}\}$ transforms the vacuum state into the state represented as

$$\begin{aligned} &\exp\{\tfrac{1}{2}A^{+}\ G_{+}(-t_0)A^{+}\}\Phi_{vac} \\ &\qquad \leftrightarrow \left\{(n!)^{1/2}\begin{pmatrix}\text{Sy}\\ \text{Asy}\end{pmatrix} g_{+}(s_1, s_2, -t_0) \cdots g_{+}(s_{n-1}, s_n, -t)\right\} \end{aligned} \tag{27.63}$$

with

$$g_{+}(s_1, s_2, -t_0) = \exp\{it_0\Omega^{s_1}\} \exp\{it_0\Omega^{s_2}\}U_{+}^{s_1}U_{+}^{s_2}\ \bar{e}(s_1, s_2). \tag{27.64}$$

Here $\bar{e}(s_1, s_2)$ is the kernel of the integral operator $\bar{E}^{s}$.

Combining the effects of the operators $\exp\{A^{+} \log U_{sc}A^{-}\}$, $\exp\{\frac{1}{2}A^{+}G(-t)A^{+}\}$ and $\exp\{\frac{1}{2}A^{+}G_{+}(-t)A^{+}\}$ we finally see from (27.59) that the state $T_{sc}^{-1}(t, t_0)\ \Phi$ is represented as

$$\Phi_w(t) = T_{sc}^{-1}(t, t_0)\Phi_w{}'(t_0) \leftrightarrow \eta \exp\{\tfrac{1}{2}i(t - t_0)h\}$$

$$\cdot\Big\{\sum_{n_0}\sum_{n_1=n_0}^{n} (n!n_0!)^{1/2}2^{n_0-n}\begin{pmatrix}\mathrm{Sy}\\ \mathrm{Asy}\end{pmatrix} g(s_n, s_{n-1}, -t)$$

$$(27.65)\qquad \cdots g(s_{n_1+2}, s_{n_1+1}, -t)g_+(s_{n_1}, s_{n_1-1}, -t_0)$$

$$\cdots g_+(s_{n_0+2}, s_{n_0+1}, -t_0)U_{sc}^{s_{n_0}} \cdots U_{sc}^{s_1}\psi_{n_0}(s)_{n_0}\Big\}.$$

This formula can be interpreted somewhat more easily if we introduce the Schrödinger states

$$(27.66)\qquad \Phi_s(t) = \exp\{-itA^+\Omega A^-\}\Phi_w(t)$$

and

$$(27.66)_0\qquad \Phi_s(t_0) = \exp\{-it_0A^+\Omega A^-\}\Phi_w(t_0).$$

The state (27.67) has the A_0-representation

$$(27.67)\qquad \Phi_s(t_0) \leftrightarrow \{\exp\{-it_0\Omega^{s_1}\} \cdots \exp\{-it_0\Omega^{s_n}\}\psi_n(s)_n\},$$

while the representation of (27.66) is derived from (27.65), (27.64), (27.62) as

$$(27.68)\qquad \Phi_s(t) \leftrightarrow \{\psi_n(s)_n(t)\},$$

with

$$\psi_n(s)_n(t) = \eta \exp\left\{\frac{1}{2}i(t - t_0)h\right\} \sum_{n_0}\sum_{n_1=n_0}^{n} (n!n_0!)^{1/2}2^{n_0-n}$$

$$(27.69)\qquad \cdot\begin{pmatrix}\mathrm{Sy}\\ \mathrm{Asy}\end{pmatrix} g(s_n, s_{n-1}) \cdots g(s_{n_1+2}, s_{n_1+1}) \exp\{i(t_0 - t)(\Omega^{n_1} + \Omega^{n_1-1})\}$$

$$\cdot g_+(s_{n_1}, s_{n_1-1}) \cdots \exp\{i(t_0 - t)(\Omega^{n_0+2} + \Omega^{n_0+1})\}g(s_{n_0+2}, s_{n_0+1})$$

$$\cdot \exp\{-it\Omega^{n_0}\}U_{sc}^{n_0} \cdots \exp\{-it\Omega^{1}\}U_{sc}^{1}\psi_{n_0}(s)_{n_0}.$$

Here the abbreviation $n_0, \cdots, 1, \cdots, n_1, \cdots, n_0 + 1$ has been used for the superscripts $s_{n_0}, \cdots, s_1, \cdots, s_{n_1}, \cdots, s_{n_0+1}$.

The functions $\psi_n(s)_n(t)$ given by (27.69) *are the* $A(t)$*-representers of the state* Φ since they are the $A(0)$-representers of the Schrödinger state $\Phi_s(t)$.

In order to interpret formula (27.69), we suppose that n_0 particles were present in the remote past and describe the amplitude of the probability of finding n particles at some time in the future as being composed of amplitudes which correspond to the following processes: *A certain number*, $n - n_1$, *of* *particles are spontaneously emitted at the final time, the* n_0 *original particles are still present but have been scattered, and the other* $n_1 - n_0$ *particles were emitted spontaneously at the original time and have then undergone a half-scattering process.*

This asymptotic description of the field scattering suffers from a certain incongruity. In deriving formula (27.47) we assumed that the operator $T^{-1}_{so}(t, t_0)$ was to be applied on the interaction state $\Phi_w(t_0)$ at the time t_0 . If this state $\Phi_w(t_0)$ approached a limit state as $t \rightarrow \infty$ (as it usually does in single particle scattering problems) it would be justified to substitute this limit state for the state $\Phi_w(t_0)$. In the present problem, however, no such limit state exists.

Nevertheless, in one typical situation, namely, when the disturbance Q is absent up to the initial time and is only switched on at this time, the asymptotic description (24.47) can be used, provided the particles composing the field in the initial state are far removed from the region in which the disturbance is noticeable. The arguments that led to the simplified form (24.47) of the transition amplitudes then remain valid.

From the fact that the amplitudes of the three different types of particles occurring in formula (27.69) have different time-dependent phase factors, $\exp\{-i(t - t_0)\Omega\}$, $\exp\{-it\Omega\}$, 1, one can deduce: *At the final time those particles are farthest removed from the disturbance region which were spontaneously emitted originally and then half scattered; next come the fully scattered original particles, while the unscattered spontaneously emitted particles are not separated from the disturbance region.*

Another consequence of the difference of these phase factors is that the contributions of the three types of particles to the *transition probabilities* are asymptotically independent of each other as $t \rightarrow \infty$. In other words, the classical rule of composition of probabilities holds asymptotically. Although it could easily be done, we shall not calculate these asymptotic transition probabilities.

28. *A Modified Electron-Positron Field*

The general theory developed in the preceding sections is valid for fermion as well as boson fields, but has so far been applied only to boson fields. As an application to fermion fields we now consider a field of electrons and positrons in the sense of Dirac under the influence of external forces which cause a linear homogeneous transformation of the annihilation and creation operators. Electromagnetic forces produced by an unquantized electromagnetic field have this property. The modification of the electron-positron field caused by such forces includes as a special case the "polarization of the vacuum." Our theory would allow us to determine this modification explicitly if the external forces satisfied the severe conditions under which the canonical transformation exists. These conditions, however, are satisfied only for special external electromagnetic fields, for example, for time-independent purely electric fields. Thus it is clear that the infinities which have been found in the investigation of the vacuum polarization are not just caused by the perturbation approximation but are inherent in the problem.

Dirac Electron

To describe the field of electrons and positrons by the process of biquantization we have first to describe *the operators formerly associated with a single Dirac electron.*

Dirac's theory of the electron was developed in connection with the requirements of the principle of relativity. The customary presentations of this theory naturally emphasize this connection; it will, however, not be apparent in the subsequent treatment. We shall select a definite Lorentz frame and employ a complete set of observables which depends on this frame. Moreover, we shall confine ourselves to external forces which do not depend on the time defined with reference to the selected frame. In principle, our procedure could be carried out also for external forces which do depend on the time (see the remarks made at the end of the subsection on "Direct Method" in Section 27); for, in this case it should be possible to cast the results into a form which exhibits invariance under Lorentz transformation. We shall not attempt to pursue this idea in the present exposition. (See a forthcoming paper by B. Zumino.)

In the following we shall deviate from our custom of denoting operators by capital letters and their eigenvalues by corresponding lower case letters. The basic single particle operators will be denoted by small letters while their eigenvalues will be indicated by primes. Thus we shall come closer to the customary notation.

An important feature of Dirac's electron is that the three components x_1, x_2, x_3 of the position x do not form a complete set of observables sufficient for its description. Additional observables are needed. Among possible additional observables one should first mention the three components of the spin $\frac{1}{2}\hbar\sigma_1$, $\frac{1}{2}\hbar\sigma_2$, $\frac{1}{2}\hbar\sigma_3$. We prefer to work with the *spin signatures* σ_1, σ_2, σ_3 and—as before—set $\hbar = 1$. The corresponding operators satisfy the relations

$$\sigma_\lambda\sigma_\nu + \sigma_\nu\sigma_\lambda = \begin{cases} 2\delta_{\lambda\nu} = 2 & \text{if} \quad \lambda = \nu \\ 0 & \text{if} \quad \lambda \neq \nu \end{cases} \tag{28.1}$$

and

$$\sigma_1\sigma_2 = i\sigma_3, \qquad \sigma_2\sigma_3 = i\sigma_1, \qquad \sigma_3\sigma_1 = i\sigma_2. \tag{28.2}$$

The triple $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ transforms like a vector when $x = (x_1, x_2, x_3)$ undergoes a rotation. There is a second set of observables β_1, β_2, β_3 which satisfy similar relations, but are not affected by a rotation of x. It is stipulated that the observables σ commute with β_1, β_2, β_3 and that all of them commute with x and p. It is further stipulated that x together with one of the variables σ and one of the variables β form a complete set.

The energy operator proposed by Dirac can be written in the form

$$\begin{aligned} H_0 &= \beta_1(\sigma_1p_1 + \sigma_2p_2 + \sigma_3p_3) + \mu\beta_3 \\ &= \beta_1(\sigma p) + \mu\beta_3 \end{aligned} \tag{28.3}$$

in which $p = (p_1, p_2, p_3)$—instead of k—is the momentum, μ the rest mass, and β a particular one among the observables $\beta_1, \beta_2, \beta_3$, say $\beta = \beta_3$. The observables $\beta_1\sigma_1, \beta_1\sigma_2, \beta_1\sigma_3$ are customarily denoted by $\alpha_1, \alpha_2, \alpha_3$.

As the representing observables we select the position x, the observable β, and—as customary—the spin signature σ_3. Their eigenvalues, the "quantum variables", will be denoted by x', β', σ'—without subscript. The representer of a state Ψ of the electron will be denoted by ψ, so that

$$\Psi \leftrightarrow \psi(x', \sigma', \beta'). \tag{28.4}$$

The unit form is

$$(\Psi, \Psi) = \sum_{\sigma'\beta'} \int |\psi(x', \sigma', \beta')|^2 \, dx'. \tag{28.5}$$

The eigenvalues of σ_3 and β are $\sigma' = \pm 1$, $\beta' = \pm 1$, and each value is assumed once. Hence we may also write

$$\Psi \leftrightarrow \{\psi(x', 1, 1), \psi(x', 1, -1), \psi(x', -1, 1), \psi(x', -1, -1)\} \tag{28.4$'$}$$

and

$$\begin{aligned}(\Psi, \Psi) = \int [|\psi(x', 1, 1)|^2 &+ |\psi(x', 1, -1)|^2 \\ &+ |\psi(x', -1, 1)|^2 + |\psi(x', -1, -1)|^2] \, dx'.\end{aligned} \tag{28.5$'$}$$

It is customary to use a subscript running from 1 to 4 instead of the four pairs (1, 1), (1, −1), (−1, 1), (−1, −1), but the latter notation seems somewhat more suitable for our purposes. Aside from this point our treatment is similar to that of Dirac [2].

The representation (28.4) can be so chosen that the matrix elements of $\sigma_1, \sigma_2, \sigma_3, \beta, \beta_1$ are

$$\begin{aligned}(\sigma' \mid \sigma_1 \mid \sigma'') &= \frac{1}{2}(1 - \sigma'\sigma'') \\ (\sigma' \mid \sigma_2 \mid \sigma'') &= \frac{i}{2}(\sigma' - \sigma'') \\ (\sigma' \mid \sigma_3 \mid \sigma'') &= \frac{1}{2}(\sigma' + \sigma''),\end{aligned} \tag{28.6}$$

$$\begin{aligned}(\beta' \mid \beta \mid \beta'') &= \frac{1}{2}(\beta' + \beta'') \\ (\beta' \mid \beta_1 \mid \beta'') &= \frac{1}{2}(1 - \beta'\beta'').\end{aligned} \tag{28.7}$$

This description is, of course, equivalent to the more customary description

$$(28.6)' \qquad \sigma_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \sigma_3 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(28.7)' \qquad \beta \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \beta_1 \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the x-representer of the momentum is $-i\nabla_x$, the (x, β, σ_3)-representation of the state $H\Psi$ is

$$(28.8) \qquad \begin{aligned} H_0\Psi &\leftrightarrow H_0^{x,\sigma_3,\beta}\psi(x', \sigma', \beta') \\ \sum_{\beta'',\sigma''} &= [-i(\beta' \mid \beta_1 \mid \beta'')(\sigma' \mid \sigma\nabla_{x'} \mid \sigma'') + \mu(\beta' \mid \beta \mid \beta'')]\psi(x', \sigma'', \beta''). \end{aligned}$$

The potential V of an external force will in general be a function of x and of $\alpha = \beta_1\sigma$. The canonical transformation, however, will not exist for all such forces, and we therefore assume a more general type of force with a potential whose representer is an integral operator with a sufficiently smooth kernel. The representer of this kernel will be denoted by $V^{x,\sigma_3,\beta}$. The modified energy operator is then

$$(28.9) \qquad H_{\text{mod}} = H_0 + V$$

and its representer

$$(28.9)' \qquad H_{\text{mod}}^{x,\sigma_3,\beta} = H_0^{x,\sigma_3,\beta} + V^{x,\sigma_3,\beta}.$$

Transformation of the Quantum Variables

Instead of the position x we may introduce the momentum p by applying the Fourier transformation to the representer $\psi(x', \sigma', \beta')$. Thus p, σ_3, β form a complete commuting system. Instead of σ_3 we may introduce the variable

$$(28.10) \qquad \tau = (\sigma_1 p_1 + \sigma_2 p_2 + \sigma_3 p_3) \mid p \mid^{-1} = (\sigma p) \mid p \mid^{-1},$$

the component of the spin signature σ in the direction of the momentum p. It is immediately seen that this observable has also the eigenvalues $\tau' = \pm 1$, each assumed once. Thus p, τ, β form a complete commuting set. Note that the undisturbed energy H_0 can be written in the form

$$(28.11) \qquad H_0 = \tau \mid p \mid \beta_1 + \mu\beta$$

and since $\tau^2 = \beta_1^2 = \beta^2 = 1$ we see that

$$H_0^2 = \mid p \mid^2 + \mu^2.$$

Introducing the "absolute undisturbed energy"

$$(28.12) \qquad \Omega = [\mid p \mid^2 + \mu^2]^{1/2}$$

and the "sign of the undisturbed energy"

$$\epsilon = H_0\Omega^{-1}, \tag{28.13}$$

we may write H_0 in the form

$$H_0 = \epsilon\Omega. \tag{28.14}$$

Clearly, the eigenvalues of ϵ are $\epsilon' = \pm 1$, each being assumed once. The undesirable feature that the energy may be in a state with a negative energy can be eliminated by considering a field composed of electrons and positrons, as will be discussed below.

We may introduce the sign of the energy ϵ as observable instead of β and thus represent the state Ψ by a function $\phi(p', \tau', \epsilon')$,

$$\Psi \leftrightarrow \phi(p', \tau', \epsilon') \tag{28.15}$$

such that

$$(\Psi, \Psi) = \sum_{\epsilon', \tau'} \int | \phi(p', \tau', \epsilon') |^2 \, dp'. \tag{28.16}$$

The transformation of the representers ϕ into the representers ψ could be given explicitly, cf. e.g. [43]; it is of the form

$$\psi(x', \sigma', \beta') = \sum_{\epsilon', \tau'} \int (4\pi)^{-3/2} \exp \{ix'p'\} \cdot (p', \sigma', \beta' \mid p', \tau', \epsilon')\phi(p', \tau', \epsilon') \, dp' \tag{28.17}$$

with

$$x'p' = x_1'p_1' + x_2'p_2' + x_3'p_3' ,$$

but we shall not use this explicit expression. We are satisfied with writing the transformation (28.17) in the form

$$\psi(x', \sigma', \beta') = \sum_{\epsilon'} R^{p;\tau}_{x,\sigma_3,\beta}(\epsilon')\phi(p', \tau', \epsilon') \tag{28.18}$$

in which the terms $R^{p;\tau}_{x,\sigma,\beta}$ ($\pm$ 1) stand for two transformations which transform functions of p', τ' into functions of x, σ, β. The variable ϵ is treated differently because it will play a special role later on.

Clearly, the transformation (28.18) gives a spectral transformation of the undisturbed energy H_0 ; for, evidently,

$$H_0^{x,\sigma_3,\beta}\psi(x', \sigma', \beta') = \sum_{\epsilon'} R^{p;\tau}_{x,\sigma,\beta}(\epsilon')\epsilon'\omega'\phi(p', \tau', \epsilon') \tag{28.19}$$

with $\omega' = [\mid p' \mid^2 + \mu^2]^{1/2}$.

Let us assume that a spectral transformation of the disturbed operator $H_{\text{mod}} = H_0 + V$ can also be constructed. We need not assume that the operator H_{mod} can be obtained from H_0 by a canonical transformation; we need not even assume that the spectrum of H_{mod} is of the same type as that of H_0. We do assume, however, that variables $\hat p = (\hat p_1, \hat p_2, \hat p_3)$, $\hat\tau$, $\hat\epsilon$ of the same type as p, τ, ϵ exist such that H_{mod} is of the form

$$H_{\text{mod}} = \hat\epsilon \Upsilon \tag{28.20}$$

in which Υ is a non-negative operator commuting with $\hat\epsilon$. Actually, this assumption is not a restriction; moreover, the variables $\hat p$, $\hat\tau$ drop out from our final formulas. The assumption was made only to enable us to apply the procedure developed in the preceding sections without modification.

The eigenvalues of the observables $\hat p$, $\hat\tau$, $\hat\epsilon$ will be denoted by p', τ', ϵ' or p'', τ'', ϵ'' without showing the "roof" explicitly. This simplification cannot cause confusion since the eigenvalues of the roofed and unroofed observables run over the same spectrum.

The $(\hat p, \hat\tau, \hat\epsilon)$-representers of the state Ψ will be denoted by $\hat\phi(p', \tau', \epsilon')$ and the transformation of these representers into $\psi(x, \sigma_3, \beta)$ will be denoted by

$$\psi(x', \sigma_3', \beta') = \sum_{\hat\epsilon'} \hat R^{\hat p;\hat\tau}_{x,\sigma_3,\beta}(\epsilon')\hat\phi(p', \tau', \epsilon'). \tag{28.21}$$

This transformation need not diagonalize the modified energy $H_o + V$; it is sufficient that the relation

$$(H_o^{x,\sigma_3,\beta} + V^{x,\sigma_3,\beta})\psi(x, \sigma_3, \beta) = \sum_{\epsilon'} \hat R^{\hat p;\hat\tau}_{x,\sigma_3,\beta}(\epsilon')\epsilon'\Upsilon^{\hat p,\hat\tau,\hat\epsilon}\hat\phi(p', \tau', \epsilon') \tag{28.22}$$

holds.

The inverse transformation of $R^{\hat p;\hat\tau}_{x,\sigma_3,\beta}(\epsilon')$ will be denoted by

$$R^{x;\sigma_3,\beta}_{\hat p,\hat\tau}(\epsilon')$$

so that

$$R^{x;\sigma_3,\beta}_{\hat p;\hat\tau}(\epsilon')R^{\hat p;\hat\tau}_{x,\sigma_3,\beta}(\epsilon'') = \delta(\epsilon' - \epsilon'') = \tfrac{1}{2}(1 + \epsilon'\epsilon''). \tag{28.23}$$

In the following we actually need only the transformation of the (p, τ, ϵ)-representers into the $(\hat p, \hat\tau, \hat\epsilon)$-representers. This transformation will be described as

$$\hat\phi(p', \tau', \epsilon') = \sum_{\epsilon''} U^-(\epsilon', \epsilon'')\phi(p', \tau', \epsilon''). \tag{28.24}$$

The operators $U^-(\pm1, \pm1)$ act on functions of p', τ' and produce functions of the same variables. Evidently, these operators are given by

$$U^-(\epsilon', \epsilon'') = R^{x;\sigma_3,\beta}_{\hat p;\hat\tau}(\epsilon')R^{p;\tau}_{x,\sigma_3,\beta}(\epsilon''). \tag{28.25}$$

By $U^+(\epsilon', \epsilon'')$ we denote the complex conjugate of the operator $U^-(\epsilon', \epsilon'')$,

$$U^+(\epsilon', \epsilon'') = \overline{U^-(\epsilon', \epsilon'')}. \tag{28.26}$$

Since the adjoint of an operator is the conjugate of its Hermitian adjoint, we find

$$U^+(\epsilon', \epsilon'') = R^{x;\sigma_3,\beta}_{p,\tau}(\epsilon'')\hat R^{\hat p;\hat\tau}_{x,\sigma_3,\beta}(\epsilon') \tag{28.25'}$$

and

$$(28.26)' \qquad {}'U^{-}(\epsilon', \epsilon'') = \overline{{}'U^{+}(\epsilon', \epsilon'')}.$$

The identities

$$(28.27) \qquad \begin{aligned} \sum_{\epsilon'''} {}'U^{\pm}(\epsilon''', \epsilon')U^{\mp}(\epsilon''', \epsilon'') &= \delta(\epsilon' - \epsilon'') \\ \sum_{\epsilon'''} U^{\mp}(\epsilon', \epsilon'''){}'U^{\pm}(\epsilon'', \epsilon''') &= \delta(\epsilon' - \epsilon''). \end{aligned}$$

can also be verified.

Electron-Positron Field

The process of biquantization could be performed by introducing a field whose states can be represented in the form

$$\Phi \leftrightarrow \{\phi_n(p, \tau, \epsilon)_n\},$$

by functions $\phi_n(p, \tau, \epsilon)_n$ of the n triples $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$. Here we omit the primes since the variables carry subscripts.

The value of ϕ_n is the amplitude of the probability that there are n particles with the momenta, spin components, and energy signs $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$ respectively. Instead of ϕ_n, one could introduce representers $\psi_n(x, \sigma_3, \beta)_n$ obtained from $\phi_n(p, \tau, \epsilon)_n$ by applying the transformation $R^{p;\tau}_{x;\sigma_3,\beta}(\epsilon)$ on ϕ_n as functions of each of the triples $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$. The corresponding annihilation operators would then be transformed in the same way. Denoting the annihilation operators associated with the (p, τ, ϵ)-representation by $A^{-}_{\epsilon 1}(p', \tau', \epsilon')$ and those associated with the (x, σ_3, β)-representation by $A^{-}_{\epsilon 1}(x', \sigma', \beta')$, we would have

$$(28.28) \qquad A^{-}_{\epsilon 1}(x', \sigma', \beta') = \sum_{\epsilon'} R^{p;\tau}_{x;\sigma_3,\beta}(\epsilon')A^{-}_{\epsilon 1}(p', \tau', \epsilon').$$

The change of these operators in time would then be the same as that of the particle representers,

$$(28.29) \qquad A^{-}_{\epsilon 1}(p', \tau', \epsilon', t) = \exp\{-it\epsilon'\omega'\}A^{-}_{\epsilon 1}(p', \tau', \epsilon')$$

with $\omega' = [\,|\,p'\,|^2 + \mu^2]^{1/2}$ and

$$(28.30) \qquad A^{-}_{\epsilon 1}(x', \sigma', \beta', t) = \sum_{\epsilon'} R^{p;\tau}_{x;\sigma_3,\beta}(\epsilon') \exp\{-it\epsilon'\omega'\}A^{-}_{\epsilon 1}(p', \tau', \epsilon').$$

The latter quantity evidently satisfies the differential equation

$$(28.31) \qquad i\nabla_t A^{-}_{\epsilon 1}(x', \sigma', \beta', t) = H_0^{x,\sigma_3,\beta}A^{-}_{\epsilon 1}(x', \sigma', \beta', t).$$

Formulas for creation operators would be obtained by taking the complex conjugates of these relations.

Actually, one performs the process of biquantization in a different way. One describes the field as consisting of electrons and positrons, both with the positive energy $\omega = [|\, p\, |^2 + \mu^2]^{1/2}$ and represents the states of the field in the form

$$\Phi \leftrightarrow \{\chi_n(p,\ \tau,\ \epsilon)_n\} \tag{28.32}$$

in terms of functions χ_n of $p_1, \tau_1, \epsilon_1, \cdots, p_n, \tau_n, \epsilon_n$ whose significance, however, differs from that of the functions ϕ_n. The function $\chi_1(p_1, \tau_1, 1)$ is the amplitude of the probability that there is one electron with momentum p_1 and spin component τ_1, while $\chi_1(p_1, \tau_1, -1)$ is the probability amplitude for the presence of one positron with momentum $-p_1$ and spin component $-\tau_1$. Similarly, $\chi_n(p,\ \tau,\ \epsilon)_n$ is the probability of the presence of n_+ electrons and n_- positrons with momenta $\pm p_1, \cdots, \pm p_n$, and spin components $\pm\tau_1, \cdots, \pm\tau_n$, if n_+ of the variables $\epsilon_1, \cdots, \epsilon_n$ have the value $+1$ while n_- of these variables have the value -1. Of course, the functions $\chi_n(p,\ \tau,\ \epsilon)_n$ should be *antisymmetric* in $(p_1, \tau_1, \epsilon_1), \cdots, (p_n, \tau_n, \epsilon_n)$.

Annihilation and creation operators of electrons will be denoted by $A^{\mp}(p',\ \tau',\ 1)$, those of positrons by $A^{\mp}(p',\ \tau',\ -1)$. Note that the time variation of the electron operators is as before

$$A^{\mp}(p',\ \tau',\ 1,\ t) = \exp\ \{\mp it\omega'\} A^{\mp}(p',\ \tau',\ 1); \tag*{(28.33)$_+$}$$

the time variation of the positron annihilation and creation operators

$$A^{\mp}(p',\ \tau',\ -1,\ t) = \exp\ \{\mp it\omega'\} A^{\mp}(p',\ \tau',\ -1), \tag*{(28.33)$_-$}$$

however, agrees with the time variation of the creation and annihilation operators $A_{\epsilon 1}^{\pm}$ of electrons with negative energy. For this reason, the positron creation operator $A^+(p',\ \tau',\ -1)$ is substituted for the annihilation operator $A_{\epsilon 1}^-(p', \tau', -1)$ in the right members of formulas (28.28, .30), while $A^-(p',\ \tau',\ 1)$ is substituted for $A_{\epsilon 1}^-(p',\ \tau',\ 1)$. The resulting expressions are then no longer pure annihilation operators. Customarily, the letter ψ is used to denote these expressions but, since we have used this letter to denote particle representers we shall use the letter Ξ, or rather Ξ^-, for this purpose. Accordingly, we introduce the "field quantities"

$$\Xi^-(x',\ \sigma',\ \beta') = \sum_{\epsilon'} R^{p;\tau}_{x;\sigma_3,\beta}(\epsilon') A^{-\epsilon'}(p',\ \tau',\ \epsilon') \tag*{(28.34)$^-$}$$

and

$$\Xi^-(x',\ \sigma',\ \beta',\ t) = \sum_{\epsilon'} R^{p;\tau}_{x;\sigma_3,\beta}(\epsilon')\ \exp\ \{-it\epsilon'\omega'\} A^{-\epsilon'}(p',\ \tau',\ \epsilon'). \tag*{(28.35)$^-$}$$

Here we have used the notation $A^- = A^{-1}$, $A^+ = A^{+1}$ which we find convenient.

The quantity Ξ^- evidently satisfies the same differential equation as $A_{\epsilon 1}^-$, see (28.30), namely

$$i\nabla_t \Xi^-(x',\ \sigma',\ \beta',\ t) = H_0^{x,\sigma_3,\beta}\Xi^-(x',\ \sigma',\ \beta',\ t). \tag*{(28.36)$^-$}$$

The adjoint Ξ^+ of the operator Ξ^- is given by

$$(28.34)^+ \qquad \Xi^+(x', \sigma', \beta') = \sum_{\epsilon'} \overline{R}^{p;\tau}_{x,\sigma_3,\beta}(\epsilon') A^{\epsilon'}(p', \tau', \epsilon');$$

its time continuation is governed by the complex conjugate of the differential equation $(28.36)^-$.

A Modified Electron-Positron Field

The process of biquantization can also be carried out with respect to the modified energy operator $H_{\text{mod}} = H_0 + V$. A field of modified electrons and positrons can be introduced accordingly. The associated annihilation and creation operators will be denoted by $B^{\mp}(p', \tau', \epsilon')$. Field quantities Ξ^-, Ξ^+ can be introduced by the relation

$$(28.37)^- \qquad \Xi^-(x', \sigma', \beta') = \sum_{\epsilon'} \hat{R}^{\hat{p};\hat{\tau}}_{x,\sigma_3,\beta}(\epsilon') B^{-\epsilon'}(p', \tau', \epsilon')$$

and its adjoint. The time variation of this operator is then given by formulas similar to (28.33); they satisfy the differential equation

$$(28.38)^- \qquad i\nabla_t \Xi^-(x', \sigma', \beta', t) = H^{x;\sigma_3,\beta}_{\text{mod}} \Xi^-(x', \sigma', \beta', t).$$

Undisturbed annihilation and creation operators $A^{\mp}$ may now again be introduced by the formula

$$(28.39)^- \qquad \Xi^-(x', \sigma', \beta', t) = \sum_{\epsilon'} R^{p;\tau}_{x,\sigma_3,\beta}(\epsilon') A^{-\epsilon'}(p', \tau', \epsilon', t)$$

and its conjugate; but the time variation of $A^{\mp}$ is no longer given by (28.33).

The *identification* of the field quantity Ξ^- expressed in terms of the modified and unmodified annihilation and creation operators $B^{\mp}$ and $A^{\mp}$ as given by formulas (28.37) and (28.39) furnishes the relationship between the two types of biquantization, that in terms of modified electrons and positrons and that in terms of unmodified ones.

Linear Transformation of Operators A into Operators B

We can eliminate the quantity Ξ^- by applying the inverse of the transformation $\hat{R}^{\hat{p};\hat{\tau}}_{x;\sigma_3,\beta}$ on both sides of relations $(28.37)^-$ and $(28.39)^-$. In view of formula (28.25) we obtain the relation

$$(28.40)^- \qquad B^{-\epsilon'}(p', \tau', \epsilon') = \sum_{\epsilon''} U^-(\epsilon', \epsilon'') A^{-\epsilon''}(p', \tau', \epsilon'').$$

We recall that each operator $U^-(\epsilon', \epsilon'')$ transforms functions of p', τ' into functions of the same variables. Taking the adjoint of relation $(28.40)^-$ we obtain

$$(28.40)^+ \qquad B^{\epsilon'}(p', \tau', \epsilon') = \sum_{\epsilon''} U^+(\epsilon', \epsilon'') A^{\epsilon''}(p', \tau', \epsilon'').$$

Both relations can be combined into

$$B^{\mp}(p', \tau', \epsilon') = \sum_{\epsilon''} U^{\mp\epsilon'}(\epsilon', \epsilon'')A^{\mp\epsilon'\epsilon''}(p', \tau', \epsilon'')$$

or into

$$\begin{aligned} B^{\mp}(p', \tau', \epsilon') = {} & \sum_{\epsilon''} \tfrac{1}{2}(1 \pm \epsilon'\epsilon'')U^{\mp\epsilon'}(\epsilon', \epsilon'')A^{-}(p', \tau', \epsilon'') \\ & + \sum_{\epsilon''} \tfrac{1}{2}(1 \mp \epsilon'\epsilon'')U^{\mp\epsilon'}(\epsilon', \epsilon'')A^{+}(p', \tau', \epsilon''). \end{aligned} \tag{28.41}$$

The latter relation is *a linear homogeneous transformation of annihilation and creation operators* of the form $\mathcal{B} = \mathcal{Y}\mathcal{A}$ treated in Section 23.

The coefficients $Y_{\pm\pm}$ of the pseudo-operator $\mathcal{Y}$ are operators acting on functions of p', τ', ϵ'. We describe them by their matrix elements $Y_{\pm\pm}(\epsilon', \epsilon'')$ with respect to the variable ϵ. These matrix elements are then operators acting on functions of p', τ'. We find

$$\begin{aligned} (\epsilon' \mid Y_{--} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')U^{-\epsilon'}(\epsilon', \epsilon''), \\ (\epsilon' \mid Y_{++} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')U^{\epsilon'}(\epsilon', \epsilon''), \end{aligned} \tag{28.42}$$

$$\begin{aligned} (\epsilon' \mid Y_{-+} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')U^{-\epsilon'}(\epsilon', \epsilon''), \\ (\epsilon' \mid Y_{+-} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')U^{\epsilon'}(\epsilon', \epsilon''). \end{aligned} \tag{28.43}$$

It is immediately seen that the matrix $\mathcal{Y}$ so given is pseudo-Hermitian. In order to verify that it satisfies the relations $'\mathcal{Y}\mathcal{Y} = 1$ and $\mathcal{Y}'\mathcal{Y} = 1$ we determine the pseudo-operator $'\mathcal{Y}$ from (24.9) and (28.25), (28.26)$'$.

$$\begin{aligned} (\epsilon' \mid {}'Y_{++} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')'U^{\epsilon''}(\epsilon'', \epsilon'), \\ (\epsilon' \mid {}'Y_{--} \mid \epsilon'') &= \tfrac{1}{2}(1 + \epsilon'\epsilon'')'U^{-\epsilon''}(\epsilon'', \epsilon'), \end{aligned} \tag*{(28.42)$'$}$$

$$\begin{aligned} (\epsilon' \mid {}'Y_{-+} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')'U^{-\epsilon''}(\epsilon'', \epsilon'), \\ (\epsilon' \mid {}'Y_{+-} \mid \epsilon'') &= \tfrac{1}{2}(1 - \epsilon'\epsilon'')'U^{\epsilon''}(\epsilon'', \epsilon'). \end{aligned} \tag*{(28.43)$'$}$$

Using relation (28.27) we then verify

$$\begin{aligned} \sum_{\epsilon'''} [(\epsilon' \mid {}'Y_{++} \mid \epsilon''')(\epsilon''' \mid Y_{--} \mid \epsilon'') + (\epsilon' \mid {}'Y_{-+} \mid \epsilon''')(\epsilon''' \mid Y_{+-} \mid \epsilon'')] \\ = \delta(\epsilon' - \epsilon''), \end{aligned}$$

$$\sum_{\epsilon'''} [(\epsilon' \mid {}'Y_{++} \mid \epsilon''')(\epsilon''' \mid Y_{-+} \mid \epsilon'') + (\epsilon' \mid {}'Y_{-+} \mid \epsilon''')(\epsilon''' \mid Y_{++} \mid \epsilon'')] = 0$$

and the other relations that constitute the relations $'\mathcal{Y}\mathcal{Y} = \mathcal{Y}'\mathcal{Y} = 1$.

As a consequence of these facts, the theory developed in the preceding sections is applicable.

Modified Vacuum State

Among the various quantities which can be calculated from the elements of the matrices $\mathcal{Y}$ and $'\mathcal{Y}$ we select the operator G which is needed to describe the modified vacuum state

$$\Phi^{B}_{\text{vac}} = \eta \exp{[-\mathcal{G}]}\Phi^{A}_{\text{vac}} .$$

Since, as seen from (28.42), the matrix elements of the operator Y_{--} can be written in the form

$$(\epsilon' \mid Y_{--} \mid \epsilon'') = \delta(\epsilon' - \epsilon'')U^{-\epsilon'}(\epsilon', \epsilon'), \tag{28.44}$$

the matrix elements of its inverse are

$$(\epsilon' \mid Y_{--}^{-1} \mid \epsilon'') = \delta(\epsilon' - \epsilon'')[U^{-\epsilon'}(\epsilon', \epsilon')]^{-1}. \tag{28.45}$$

Since the matrix elements of Y_{-+} can be written as

$$(\epsilon' \mid Y_{-+} \mid \epsilon'') = \delta(\epsilon' + \epsilon'')U^{-\epsilon'}(\epsilon', -\epsilon'),$$

see (28.43), those of the operator $G = Y_{--}^{-1}\, Y_{-+}$ are

$$(\epsilon' \mid G \mid \epsilon'') = \delta(\epsilon' + \epsilon'')[U^{-\epsilon'}(\epsilon', \epsilon')]^{-1}U^{-\epsilon'}(\epsilon', -\epsilon'). \tag{28.46}$$

It is to be noted that this operator does not depend on the choice of the artificial variables $\hat{p}$, $\hat{\tau}$; for, the operators $U^{\mp}(\epsilon', \epsilon'')$ transform (p, τ)-representers into $(\hat{p}, \hat{\tau})$-representers and hence the inverse $[U^{-\epsilon'}(\epsilon', \epsilon')]^{-1}$ transforms $(\hat{p}, \hat{\tau})$-representers into (p, τ)-representers. The arguments of the kernel

$$(p', \tau', \epsilon' \mid G \mid p'', \tau'', \epsilon'')$$

of the operator G are therefore the original variables associated with the undisturbed field. The antisymmetry of this function is guaranteed by the general theory.

Note that the kernel G vanished unless $\epsilon' = -\epsilon''$. In other words, *the modified vacuum state consists of pairs of electrons and positrons.* For example, the amplitude of the probability that there is exactly one such pair of which the first is an electron with momentum p_1 and spin component τ_1, and the second a positron with momentum $-p_2$ and spin component $-\tau_2$ is

$$\sqrt{2}\, \eta(p_1 , \tau_1 , 1 \mid G \mid p_2 , \tau_2 , -1).$$

Vacuum Transition Probability

The probability that the field will be found to be in the unmodified vacuum state if it is known to be in the modified vacuum state—or vice versa—is, by (24.26), (24.45),

$$Pr\,(N, 0 \mid M, 0) = \mid \eta \mid^2 = \exp{\{\tfrac{1}{2}\, Tr \log {'Y_{++}}Y_{--}\}}.$$

The trace occurring here is easily calculated. From (28.43), (28.42)′ we determine the ϵ-matrix element of the operator $'Y_{++}Y_{--}$ as

$$(\epsilon' \mid 'Y_{++}Y_{--} \mid \epsilon'') = \tfrac{1}{4} \sum_{\epsilon'''} (1 + \epsilon'\epsilon''')(1 + \epsilon'''\epsilon'')'U^{\epsilon'''}(\epsilon''', \epsilon')U^{-\epsilon'''}(\epsilon''', \epsilon'')$$

or, since $1 + \epsilon'\epsilon'' = \delta(\epsilon' - \epsilon'')$,

$$(\epsilon' \mid 'Y_{++}Y_{--} \mid \epsilon'') = \delta(\epsilon' - \epsilon'')'U^{\epsilon'}(\epsilon', \epsilon')U^{-\epsilon'}(\epsilon', \epsilon'). \tag{28.47}$$

From this expression one derives

$$(\epsilon' \mid \log 'Y_{++}Y_{--} \mid \epsilon'') = \delta(\epsilon' - \epsilon'') \log 'U^{\epsilon'}(\epsilon', \epsilon')U^{-\epsilon'}(\epsilon', \epsilon')$$

whence

$$\begin{aligned} Tr \log 'Y_{++}Y_{--} &= \sum_{\epsilon'} Tr \log 'U^{\epsilon'}(\epsilon', \epsilon')U^{-\epsilon'}(\epsilon', \epsilon') \\ &= Tr \log 'U^{+}(1, 1)U^{-}(1, 1) + Tr \log 'U^{-}(-1, -1)U^{+}(-1, -1). \end{aligned} \tag{28.48}$$

Consequently,

$$\begin{aligned} P(N, 0 \mid M, 0) = \exp \{\tfrac{1}{2} Tr \log &'U^{+}(1, 1)U^{-}(1, 1) \\ &+ \tfrac{1}{2} Tr \log 'U^{-}(-1, -1)U^{+}(-1, -1)\}. \end{aligned} \tag{28.49}$$

Thus *the vacuum transition probability is calculated explicitly* inasmuch as the transformations $U^{\pm}$, $'U^{\pm}$ are supposed to be known.

Note that in the left member of (28.48) one should take the trace of an operator acting on a function of p', τ', ϵ' while in the two right members the traces refer to p', τ' only.

It should also be noted that the probability (28.49) is independent of the choice of the artificial variables $\hat{p}$, $\hat{\tau}$. For, the operators $U^{\pm}(\epsilon', \epsilon'')$ transform (p, τ)-representers into $(\hat{p}, \hat{\tau})$-representers, while the operators $'U^{\pm}(\epsilon', \epsilon'')$ transform $(\hat{p}, \hat{\tau})$-representers into (p, τ)-representers. It follows that the operators $'U^{\pm}U^{\mp}$ transform (p, τ)-representers into (p, τ)-representers and are independent of the choice of the variables $\hat{p}$, $\hat{\tau}$.

It is possible to give the expression (28.49) for the vacuum transition probability a very concise form which involves only the signs $\hat{\epsilon}$ and ϵ of the modified and unmodified energy.

Clearly, as seen from (28.24), (28.27)′, the ϵ-matrix of the operator $\hat{\epsilon}$ is

$$(\epsilon' \mid \hat{\epsilon} \mid \epsilon'') = \sum_{\epsilon'''} 'U^{+}(\epsilon''', \epsilon')\epsilon'''U^{-}(\epsilon''', \epsilon'').$$

The ϵ-matrix of the operator $\frac{1}{4}(\hat{\epsilon} - \epsilon)^2$ is therefore

$$\begin{aligned} (\epsilon' \mid \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2 \mid \epsilon'') &= \tfrac{1}{4}\sum_{\epsilon'''} 'U^{+}(\epsilon''', \epsilon')(\epsilon''' - \epsilon')(\epsilon''' - \epsilon'')U^{-}(\epsilon''', \epsilon'') \\ &= \delta(\epsilon' + \epsilon'')'U^{+}(-\epsilon', \epsilon')U^{-}(-\epsilon', \epsilon') \end{aligned}$$

and consequently its trace is

$$(28.50)\quad \tfrac{1}{4}\, Tr\, (\hat{\epsilon} - \epsilon)^2 = Tr\, {}'U^+(-1, 1)U^-(-1, 1) + Tr\, {}'U^+(1, -1)U^-(1, -1).$$

Similarly, we find

$$(\epsilon' \mid 1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2 \mid \epsilon'') = \delta(\epsilon' + \epsilon'')\, {}'U(\epsilon', \epsilon')U^-(\epsilon', \epsilon'),$$

$$(\epsilon' \mid \log [1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2] \mid \epsilon'') = \delta(\epsilon' + \epsilon'') \log {}'U^+(\epsilon', \epsilon')U^-(\epsilon', \epsilon')$$

and consequently

$$(28.51)\quad \begin{aligned} &Tr \log [1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2] \\ &\quad = Tr \log {}'U^+(1, 1)U^-(1, 1) + Tr \log {}'U^+(-1, -1)U^-(-1, -1). \end{aligned}$$

Thus we see that *the vacuum transition probability* (28.49) *can be written in the form*

$$(28.52)\quad P(N, 0 \mid M, 0) = \exp \{\tfrac{1}{2}\, Tr \log [1 - \tfrac{1}{4}(\hat{\epsilon} - \epsilon)^2]\}.$$

The fact that this expression is independent of the quantum variables chosen will prove useful.

If the disturbance is small so that the transformation $\mathcal{Y}$ differs little from the identity one may approximate expression (28.52) by the expression

$$(28.53)\quad P(N, 0 \mid M, 0) \sim 1 - \tfrac{1}{8}\, Tr\, (\hat{\epsilon} - \epsilon)^2.$$

In fact, the condition that the trace occurring in (28.52) be finite is equivalent with the condition that the trace $Tr\, (\hat{\epsilon} - \epsilon)^2$ occurring in (28.53) be finite (cf. the discussion in connection with (25.47), (25.49)).

The right member of (28.53) can be evaluated by (28.50). Employing the kernels $U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')$ and ${}'U^+(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')$ of $U^-(\epsilon', \epsilon'')$ and ${}'U^+(\epsilon'', \epsilon')$ we find, using (28.26)′,

$$(28.53)_1\quad \begin{aligned} P(N, 0 \mid M, 0) \sim 1 - \tfrac{1}{2} \sum_{\tau', \tau''} \iint \{ &\mid U^-(p', \tau', 1 \mid p'', \tau'', 1) \mid^2 \\ &+ \mid U^-(p', \tau', -1 \mid p'', \tau'', -1) \mid^2\}\, dp'\, dp''. \end{aligned}$$

Perturbation Approximation

We have supposed the transformations $U^{\pm}(\epsilon', \epsilon'')$ associated with a single electron to be known. Insofar as this supposition is correct we may say that all relevant quantities such as the number $\mid \eta \mid$ and the operator G can be determined without using an approximation procedure. In most actual cases, however, the determination of the operators $U^{\pm}(\epsilon', \epsilon'')$ themselves requires the use of an approximation procedure. Under favorable conditions one may employ a perturbation procedure. In typical cases it will mostly be required that the spectrum of the modified energy H_{mod} is the same as that of the unmodified energy H_0 and that, moreover, the artificial variables $\hat{p}$, $\hat{\tau}$, $\hat{\epsilon}$ can be so chosen that the modified absolute energy $\hat{\omega}$ is the same function of $\hat{p}$, $\hat{\tau}$ as the unmodified absolute energy is of p, τ, i.e.,

$$\hat{\omega} = \omega(\hat{p}, \hat{\tau}) = (|\hat{p}|^2 + \mu^2)^{1/2} \tag{28.54}$$

Let $(p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon'')$ be the kernel of the disturbing energy and $U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')$ the kernel of the operator $U^-(\epsilon', \epsilon'')$. Because of (28.54) the latter transformation gives the spectral representation of the operator $H_{\text{mod}} = H_0 + V$ and consequently the relation

$$\begin{aligned} \epsilon'\omega' U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') &= U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')\epsilon''\omega'' \\ &\quad + \int \sum_{\tau''', \epsilon'''} U^-(p', \tau', \epsilon' \mid p''', \tau''', \epsilon''') \\ &\quad \cdot (p''', \tau''', \epsilon''' \mid V \mid p', \tau', \epsilon')\, dp''' \end{aligned} \tag{28.55}$$

holds. Since

$$U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') = \delta(p' - p'', \tau' - \tau'', \epsilon' - \epsilon'') \tag{28.56}$$

in first approximation, equation (28.52) becomes in first approximation

$$\begin{aligned} &\epsilon'\omega' U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') \\ &\quad = U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'')\epsilon''\omega'' + (p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon''). \end{aligned} \tag{28.57}$$

A solution of this equation is

$$\begin{aligned} U^-(p', \tau', \epsilon' \mid p'', \tau'', \epsilon'') &= \delta(p' - p'', \tau' - \tau'', \epsilon' - \epsilon'') \\ &+ [(\epsilon'\omega' - \epsilon''\omega'')^{-1} + i\pi\, \delta(\epsilon'\omega' - \epsilon''\omega'')](p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon'') \end{aligned} \tag{28.58}$$

For $\epsilon'' = -\epsilon'$, in particular,

$$\begin{aligned} &U^-(p', \tau', \epsilon' \mid p'', \tau'', -\epsilon') \\ &\qquad = \epsilon'(\omega' + \omega'')^{-1}(p', \tau', \epsilon' \mid V \mid p'', \tau'', -\epsilon'). \end{aligned} \tag{28.59}$$

Instead of the factor $i\pi$ the factor $-i\pi$ could just as well have been chosen on the right hand side of (28.58). In either case the operator U is unitary in first approximation, cf. [2] and [44]. However, the sign of this factor is irrelevant in the present context since it does not contribute to the right member of (28.59).

The expression $Tr\ (\hat{\epsilon} - \epsilon)^2$ which occurs in the approximate expression

(28.53) for the vacuum transition probability can now, by (28.54), be approximately expressed as

$$
\begin{aligned}
1 - Pr\,(N, 0 \mid M, 0) &\sim \tfrac{1}{2} \sum_{\tau', \tau''} \iint \{ \mid p', \tau', 1 \mid V \mid p'', \tau'', -1) \mid^2 \\
&\quad + \mid (p', \tau', -1 \mid V \mid p'', \tau'', 1) \mid^2 \} (\omega' + \omega'')^{-2}\, dp'\, dp''.
\end{aligned}
\tag{28.60}
$$

This formula was found by H. E. Moses [51] who applied a perturbation procedure directly to the field equations. In the following we present a slight modification of his arguments.

It is possible to write expression (28.60) in a form which does not depend on the choice of τ, ϵ or σ_3, β as quantum variables. To this end we introduce the operator

$$
W = \tfrac{1}{4}(1 + \epsilon) V (1 - \epsilon) + \tfrac{1}{4}(1 - \epsilon) V (1 + \epsilon) = \tfrac{1}{2} V - \tfrac{1}{2} \epsilon V \epsilon. \tag{28.61}
$$

Its (p, τ, ϵ)-kernel is

$$
(p', \tau', \epsilon' \mid W \mid p'', \tau'', \epsilon'') = \delta(\epsilon' + \epsilon'')(p', \tau', \epsilon' \mid V \mid p'', \tau'', \epsilon''), \tag{28.62}
$$

and hence we have

$$
\begin{aligned}
&1 - Pr\,(N, 0 \mid M, 0) \\
&\sim \frac{1}{2} \iint \sum_{\tau', \tau''} \sum_{\epsilon', \epsilon''} \mid (p', \tau', \epsilon' \mid W \mid p'', \tau'', \epsilon'') \mid^2 (\omega' + \omega'')^{-2}\, dp'\, dp''.
\end{aligned}
\tag{28.63}
$$

Next we assign to every operator Λ acting on states Ψ of single particles its "p-kernel"; i.e. the operator $(p' \mid \Lambda \mid p'')$ which acts on "spin states" represented by functions of σ_3, β or τ, ϵ and whose (p, τ, ϵ)- and (p, σ_3, β)-representers are $(p', \tau', \epsilon' \mid \Lambda \mid p'', \tau'', \epsilon'')$ and $(p', \sigma_3', \beta' \mid \Lambda \mid p'', \sigma_3'', \beta'')$ respectively. The operator with the kernel $(p'', \tau'', \epsilon'' \mid \Lambda \mid p', \tau', \epsilon')$ will be denoted by $(p' \mid \Lambda^* \mid p'')$.

Observing that the (p, τ, ϵ)-representer of the operator $(p'' \mid W^* \mid p') \cdot (p' \mid W \mid p'')$ is $\sum_{\tau'''} \sum_{\epsilon'''} (p', \tau''', \epsilon''' \mid W \mid p'', \tau', \epsilon')(p', \tau''', \epsilon''' \mid W \mid p'', \tau'', \epsilon'')$ we see that we may write

$$
\sum_{\tau' \tau''} \sum_{\epsilon' \epsilon''} \mid (p', \tau', \epsilon' \mid W \mid p'', \tau'', \epsilon'') \mid^2 = Tr\,(p'' \mid W^* \mid p')(p' \mid W \mid p'')
$$

and hence formula (28.63) becomes

$$
\begin{aligned}
&1 - P(N, 0 \mid M, 0) \\
&\sim \frac{1}{2} \iint Tr\,(p'' \mid W^* \mid p')(p' \mid W \mid p'')(\omega' + \omega'')^{-2}\, dp'\, dp''.
\end{aligned}
\tag{28.64}
$$

The trace occurring here refers to the variables τ, ϵ; but, being independent of the choice of the variables, it could just as well be evaluated by employing the variables σ_3 , β.

This evaluation will be facilitated by using the operators $(p' \mid \epsilon \mid p'')$. From the definition (28.13) of ϵ and Ω, and the definition (28.3) of H_0 we see that the operator $(p' \mid \epsilon \mid p'')$ can be written in the form

(28.65)
$$(p' \mid \epsilon \mid p'') = \delta(p' - p'')\epsilon(p')$$

where $\epsilon(p')$ is an operator acting on spin states and depending on p'; specifically

(28.66)
$$\epsilon(p') = [\omega']^{-1}[\beta_1(\sigma p') + \mu\beta].$$

Using this operator we may derive from (28.61) the expression

(28.67)
$$2(p' \mid W \mid p'') = (p' \mid V \mid p'') - \epsilon(p')(p' \mid V \mid p'')\epsilon(p'').$$

Since $[\epsilon(p')]^2 = 1$, we have

$$\begin{aligned}4(p'' \mid W^* \mid p')(p' \mid W \mid p'') &= (p'' \mid V^* \mid p')(p' \mid V \mid p'') \\ &- \epsilon(p'')(p'' \mid V^* \mid p')\epsilon(p')(p' \mid V \mid p'') - (p'' \mid V^* \mid p')\epsilon(p')(p' \mid V \mid p'')\epsilon(p'') \\ &+ \epsilon(p'')(p'' \mid V^* \mid p')(p' \mid V \mid p'')\epsilon(p'')\end{aligned}$$

whence, since $Tr\, AB = Tr\, BA$, the formula

(28.68)
$$\begin{aligned}&2\, Tr\, (p'' \mid W^* \mid p')(p' \mid W \mid p'') \\ &= Tr\, (p'' \mid V^* \mid p')(p' \mid V \mid p'') - Tr\, \epsilon(p'')(p'' \mid V^* \mid p')\epsilon(p')(p' \mid V \mid p'')\end{aligned}$$

results; it may be used in (28.64).

Expression (28.68) can be greatly simplified in the important case in which the operator V is constant as regards spin states so that $V^{p,\sigma_3,\beta} = V^p$ acts on functions $\psi(p, \sigma_3 , \beta)$ only insofar as they depend on p. In this case the operator $\epsilon(p')$ commutes with the operator $(p' \mid V \mid p'')$ and hence we have

(28.69,
$$\begin{aligned}&2\, Tr\, (p'' \mid W^* \mid p')(p' \mid W \mid p'' \\ &= [Tr\, 1 - Tr\, \epsilon(p'')\epsilon(p')](p'' \mid V^* \mid p')(p' \mid V \mid p'').\end{aligned}$$

The trace of the identity as regards spin states is evidently

(28.70)
$$Tr\, 1 = 4.$$

The trace of $\epsilon(p'')\epsilon(p')$ is easily evaluated from (28.66) combined with the commutation relations (28.1),

(28.71)
$$Tr\, \epsilon(p'')\epsilon(p') = 4(\omega'\omega'')^{-1}[p''p' + \mu^2].$$

Hence

$$\tfrac{1}{2}\,Tr\,[1 - \epsilon(p'')\epsilon(p')] = 1 - (\omega'\omega'')^{-1}(p'p'' + \mu^2)$$

$$= (\omega'\omega'')^{-1}[\omega'\omega'' - p'p'' - \mu^2]$$

and formula (28.64) becomes

$$\begin{aligned} &1 - P(N, 0 \mid M, 0) \\ &= \iint \mid (p' \mid V \mid p'') \mid^2 [\omega'\omega'' - p'p'' - \mu^2](\omega'\omega'')^{-1}(\omega' + \omega'')^{-2}\, dp'\, dp''. \end{aligned} \tag{28.72}$$

Of particular interest is the special case where the disturbing energy V is an external potential which is a function of the position x. Then the p-kernel of V is a function of $p' - p''$,

$$(p' \mid V \mid p'') = V(p'' - p').$$

Moses observed [49] that in this case the right member of formula (28.72) is finite provided the function $V(p)$ dies out sufficiently strongly as $\mid p \mid \rightarrow \infty$. In fact, it is sufficient that $\mid V(p) \mid$ be bounded and integrable. To see this we may introduce $\tilde{p} = p'' - p'$ and p' as new variables and set

$$K(p') = (\omega')^4 \int \mid V(\tilde{p}) \mid^2 [\omega'\omega'' - p'p'' - \mu^2](\omega'\omega'')^{-1}(\omega' + \omega'')^{-2}\, d\tilde{p}, \tag{28.73}$$

so that

$$1 - P(N, 0 \mid M, 0) = \int K(p')(\omega')^{-4}\, dp'. \tag{28.74}$$

For large values of $\mid p' \mid$ we have

$$\omega'\omega'' - p'p'' - \mu^2 \sim \tfrac{1}{2} \mid p' \mid^{-2} [\mid p' \mid^2 \mid \tilde{p} \mid^2 - (p'\tilde{p})^2].$$

Here the right member remains bounded and the same is, of course, true of $(\omega')^3(\omega'')^{-1}(\omega' + \omega'')^{-2}$. Consequently, $K(p')$ is bounded for large values of $\mid p' \mid$ and hence the integral in (28.74) is finite.

This integral gives only the contribution of second order to the probability $P(N, 0 \mid M, 0)$. A closer analysis would show that the complete probability itself is also finite *provided the disturbance V is small enough*. In this case then *the canonical transformation T exists*. It is true that the expectation values of certain observables—such as the charge density at any point—are infinite in the vacuum state. One must expect such occurrences whenever the operators which correspond to the observables are unbounded. Of course, such occurrences alone do not invalidate the theory.

The situation is quite different if the disturbance V is not constant as regards spin states, for example in case

$$V = (\alpha A) = \beta_1(\sigma A)$$

where $A = \{A_1, A_2, A_3\}$ is the vector potential of an external electromagnetic field. In this case relation (28.68) leads to

$$\tfrac{1}{2}\, Tr\, (p'' \mid W^* \mid p')(p' \mid W \mid p'') = (\omega'\omega'')^{-1}[\omega'\omega'' + p'p'' + \mu^2]$$

provided A is divergence-free, $pA = 0$, which we may assume. The arguments that led to the boundedness of the expression $K(p')$ now break down and, consequently, the right member of formula (28.64) is infinite. It would seem that one should assume the field to be myriotic in this case and it may perhaps be that one could then rederive in a mathematically satisfactory way some of the finite results that have been obtained by using—in intermediate steps—certain terms which are actually infinite.

BIBLIOGRAPHY TO PART V

[42] Baker, H. F., *Alternates and continuous groups,* Proceedings of the London Mathematical Society, 2nd Series, Volume 3, 1905, pp. 24–47.

[43] Hausdorff, F., *Die symbolische Exponentialformel in der Gruppentheorie,* Berichte der Sächsischen Akademie der Wissenschaften, Mathematisch-Physikalishe Klasse, Volume 58, Leipzig, 1906, pp. 19–48.

[44] Møller, C., *General properties of the characteristic matrix in the theory of elementary particles, II,* K. Danske Videnskabernes Selskab., Volume 22, No. 19, 1946.

[45] Friedrichs, K. O., *On the perturbation of continuous spectra,* Communications on Pure and Applied Mathematics, Volume 1, 1948, pp. 361–406.

[46] Feynman, R. P., *The theory of positrons,* Physical Reviews, Volume 76, 1949, pp. 749–759.

[47] Magnus, W., *A connection between the Baker-Hausdorff formula and a problem of Burnside,* Annals of Mathematics, Volume 52, 1950, pp. 111–126.

[48] Yang, C. N., and Feldman, D., *The S-matrix in the Heisenberg representation,* Physical Reviews, Volume 79, 1950, pp. 972–978.

[49] Wick, G. C., *The evaluation of the collision matrix,* Physical Review, Volume 80, 1950, pp. 268–272.

[50] Sollfrey, W., and Goertzel, G., *Some quantum mechanical divergences of a simple field,* Physical Reviews, Volume 83, 1951, pp. 1038–1044.

[51] Moses, H. E., *The canonical transformation for an electron-positron field coupled to a time-independent electromagnetic field,* Physical Reviews, Volume 89, No. 1, 1953, pp. 115–122.

[52] Moses, H. E., *Electrons and positrons in a time-independent electromagnetic field: A solution in the Schrödinger picture,* Part II (in preparation).

APPENDIX

Lorentz Invariant Treatment of Boson Fields

The treatment of fields subject to homogeneous forces, developed in Part V, was essentially not invariant under Lorentz transformation. Nevertheless, the method developed there can be adapted to a Lorentz invariant treatment. This is done by B. Zumino (in a forthcoming paper). We shall give an account of some of his results for the case of a boson field.

29. *Unquantized Field*

It is first necessary to consider unquantized fields described by a field quantity ξ which depends on the point in space $\underset{3}{x} = \{x_1, x_2, x_3\}$ and the time $x_0 = t$. Introducing the space-time point

$$\underset{4}{x} = \{x_0, \underset{3}{x}\},$$

we consider ξ as a function $\xi = \xi(\underset{4}{x})$. We require that this function satisfy the differential equation

$$(\mu^2 - \Box)\xi + Q^x\xi = 0, \tag{29.1}$$

the "modified wave equation," in which

$$\Box = \nabla^2 - \nabla_t^2 \tag{29.2}$$

is the D'Alembertian operator and Q^x is an operator acting on functions $\xi(\underset{4}{x})$. We assume that Q^x can be described with the aid of a symmetric real kernel $(\underset{4}{x} \mid Q \mid \underset{4}{x}')$, i.e. we assume

$$Q^x\xi(\underset{4}{x}) = \int (\underset{4}{x} \mid Q \mid \underset{4}{x}')\xi(\underset{4}{x}')\, d\underset{4}{x}', \tag{29.3}$$

where $d\underset{4}{x} = dx_0 \cdots dx_3$ and the integration extends over the whole four-dimensional space. Below we shall come back to the question which types of functions may be admitted for this kernel.

Our problem concerning the wave equation (29.1) consists in establishing a solution $\xi(\underset{4}{x})$ which as $t \to -\infty$ approaches a given undisturbed wave, i.e. a solution with $Q = 0$, and in determining the undisturbed wave which it approaches as $t \to \infty$.

Momentum Representation

We shall discuss this problem in the momentum- rather than in the x-representation which is employed by Zumino. Although the use of the x-representation, which is somewhat preferred in the current literature on field quantum theory, offers various formal advantages, we prefer the treatment in the momentum representation because the problem can then be subsumed directly under the theory developed in Part V.

We introduce the four-dimensional energy-momentum, or simply momentum,

$$\underset{4}{k} = \{k_0\,,\, \underset{3}{k}\}, \qquad \underset{3}{k} = \{k_1\,,\, k_2\,,\, k_3\},$$

and connect the function $\xi(\underset{4}{x})$ with a Fourier transform, which we simply denote by $\xi(\underset{4}{k})$. We find it convenient to define the (four-dimensional) Fourier transform by the formulas

$$\xi(\underset{4}{x}) = -\theta^5 i \int \exp\{-i\,\underset{4}{x}\,\underset{4}{k}\}\xi(\underset{4}{k})\,d\underset{4}{k}, \tag{29.4}$$

$$\xi(\underset{4}{k}) = \theta^3 i \int \exp\{i\,\underset{4}{x}\,\underset{4}{k}\}\xi(\underset{4}{x})\,d\underset{4}{x}, \tag{29.4$'$}$$

in which $\theta = (2\pi)^{-1/2}$, cf. (12.25), and

$$\underset{4}{x}\,\underset{4}{k} = x_0k_0 - \underset{3}{x}\,\underset{3}{k}, \qquad \underset{3}{x}\,\underset{3}{k} = x_1k_1 + x_2k_2 + x_3k_3\,. \tag{29.5}$$

As Fourier transform of the kernel $(\underset{4}{x} \mid Q \mid \underset{4}{x'})$ we introduce the kernel

$$(\underset{4}{k} \mid Q \mid \underset{4}{k'}) = \theta^8 \iint \exp\{i\,\underset{4}{x}\,\underset{4}{k} + i\,\underset{4}{x'}\underset{4}{k'}\}(\underset{4}{x} \mid Q \mid \underset{4}{x'})\,d\underset{4}{x}\,d\underset{4}{x'} \tag{29.6}$$

which is symmetric,

$$(\underset{4}{k'} \mid Q \mid \underset{4}{k}) = (\underset{4}{k} \mid Q \mid \underset{4}{k'}), \tag{29.7}$$

just as $(\underset{4}{x} \mid Q \mid \underset{4}{x'})$ is. This symmetry is achieved by making the exponent in (29.6) symmetric. The transform of the operator Q^x is then the operator Q^k associated with the kernel $(\underset{4}{k} \mid Q \mid \underset{4}{k'})$:

$$Q^k\xi(\underset{4}{k}) = \int (\underset{4}{k} \mid Q \mid -\underset{4}{k'})\xi(\underset{4}{k'})\,d\underset{4}{k'}; \tag{29.8}$$

the function $Q^k\xi(\underset{4}{k})$ is then exactly the Fourier transform of the function $Q^x\xi(\underset{4}{x})$. In the following we omit the superscript k and write Q instead of Q^k.

The Fourier transform $\xi(\underset{4}{k})$ of a real-valued function $\xi(\underset{4}{x})$ is evidently characterized by the condition

$$\overline{\xi(\underset{4}{k})} = -\xi(-\underset{4}{k}); \tag{29.9}$$

the Fourier transform of an operator Q with a real kernel $(\underset{4}{x} \mid Q \mid \underset{4}{x'})$ is characterized by the condition

$$\overline{(\underset{4}{k} \mid Q \mid \underset{4}{k'})} = (-\underset{4}{k} \mid Q \mid -\underset{4}{k'}). \tag{29.10}$$

The Fourier transform of the differential equation (29.1) reduces to the equation

$$(\mu^2 - \underset{4}{k}^2)\xi(\underset{4}{k}) + Q\xi(\underset{4}{k}) = 0 \tag{29.11}$$

with

$$\underset{4}{k}^2 = k_0^2 - \underset{3}{k}^2 , \qquad \underset{3}{k}^2 = k_1^2 + k_2^2 + k_3^2 .$$

The solutions $\xi(k)$ will be improper functions of a certain type. In order to describe them we introduce the improper function

$$\zeta(\underset{4}{k}) = 2\pi i[\delta(k_0 - \omega) - \delta(k_0 + \omega)](2\omega)^{-1} \tag{29.12}$$

with $\omega = (\mu^2 + \underset{3}{k}^2)^{1/2}$.

It is known that this function is invariant under Lorentz transformation provided the sign of k_0 is not changed; this fact is exhibited by writing the function in the form

$$\zeta(\underset{4}{k}) = 2\pi i \operatorname{sgn} k_0 \, \delta(\underset{4}{k}^2 - \mu^2). \tag{29.13}$$

With the aid of $\zeta(\underset{4}{x})$ we form the two functions

$$\zeta^{\pm}(\underset{4}{k}) = (\underset{4}{k}^2 - \mu^2)^{-1} \pm \tfrac{1}{2}\zeta(\underset{4}{k}). \tag{29.14}$$

We denote by Z and $Z^{\pm}$ the operators which multiply a function $\phi(\underset{4}{k})$ by $\zeta(\underset{4}{k})$ and $\zeta^{\pm}(\underset{4}{k})$, respectively. Thus

$$Z\phi(\underset{4}{k}) = \zeta(\underset{4}{k})\phi(\underset{4}{k}), \tag{29.15}$$

$$Z^{\pm}\phi(\underset{4}{k}) = \zeta^{\pm}(\underset{4}{k})\phi(\underset{4}{k}). \tag{29.16}$$

As seen from (29.15), (29.16) the relation

$$Z^{+} - Z^{-} = Z \tag{29.17}$$

holds.

The $\underset{4}{x}$-kernels of the operators Z and $Z^{\pm}$ are respectively the functions

$$-\Delta(\underset{4}{x} - \underset{4}{x}'), \qquad \Delta^{\text{adv}}(\underset{4}{x} - \underset{4}{x}'), \qquad \Delta^{\text{ret}}(\underset{4}{x} - \underset{4}{x}')$$

in the notation of Schwinger, cf. [55].

We note that a function of the form $\xi(\underset{4}{k}) = Za(\underset{4}{k})$ satisfies the homogeneous equation

$$(29.18) \qquad (\mu^2 - \underset{4}{k}^2)\xi(\underset{4}{k}) = (\mu^2 - \underset{4}{k}^2)Za(\underset{4}{k}) = 0,$$

while a function of any form $\xi(\underset{4}{k}) = Z^{\pm}g(\underset{4}{k})$ satisfies the non-homogeneous equation

$$(29.19) \qquad (\mu^2 - \underset{4}{k}^2)\xi(\underset{4}{k}) = (\mu^2 - \underset{4}{k}^2)Z^{\pm}g(\underset{4}{k}) = -g(\underset{4}{k}).$$

The values of the function $a(\underset{4}{k})$ on the hyperboloid $\underset{4}{k}^2 = \mu^2$ will be referred to as the "restricted" values of $a(\underset{4}{k})$. Evidently the values of $Za(\underset{4}{k})$ depend only on these restricted values of $a(\underset{4}{k})$.

The Fourier transform (29.4) of the function $Za(\underset{4}{k})$ can be written in the form

$$(29.20) \qquad \xi(\underset{4}{x}) = \theta^3 \int a(\underset{4}{k}) \exp\{-i \underset{4}{x}\, \underset{4}{k}\} \operatorname{sgn} k_0\, \delta(\underset{4}{k}^2 - \mu^2)\, d\underset{4}{k},$$

or, as seen from (29.12), in the form

$$(29.21) \qquad \xi(\underset{4}{x}) = \theta^3 \int [a(\omega, \underset{3}{k}) \exp\{-i \underset{3}{x}\, \underset{3}{k} - it\omega\}$$

$$- a(-\omega, -\underset{3}{k}) \exp\{it\omega - i \underset{3}{x}\, \underset{3}{k}\}]\, d\underset{3}{k}(2\omega)^{-1},$$

after having substituted $-\underset{3}{k}$ for $\underset{3}{k}$ in the second contribution to the last formula. Clearly, this function $\xi(\underset{4}{x})$ is a solution of the undisturbed wave equation $(\mu^2 - \Box)\xi(\underset{4}{x}) = 0$ and hence represents the wave quantity of a free field. It is evident from (29.21) that this quantity $\xi(\underset{4}{x})$ depends only on the restricted values of the amplitudes $a(\underset{4}{k})$.

The Fourier transform of the function $Z^{-}g(\underset{4}{k})$ approaches zero as $t \to -\infty$, while that of $Z^{+}g(\underset{4}{k})$ approaches zero as $t \to \infty$, provided the function is sufficiently regular; for related statements see e.g. [45]. It is because of this property that we work with the operators $Z^{\pm}$ and not with other related operators.

Equations for Wave Amplitudes

In view of the preceding remarks, it is natural to assume that the solution $\xi(\underset{4}{k})$ of equation (29.11) is of the form

$$\xi(\underset{4}{k}) = Za(\underset{4}{k}) + Z^{-}g(\underset{4}{k}), \tag{29.22}$$

and to interpret the function $a(\underset{4}{k})$ as the amplitude of the "incoming" free field, cf. (29.20) Similarly, one may assume $\xi(\underset{4}{x})$ to be of the form

$$\xi(\underset{4}{k}) = Zb(\underset{4}{k}) + Z^{+}g(\underset{4}{k}) \tag{29.23}$$

and interpret $b(\underset{4}{k})$ as the amplitude of the "outgoing" free field. From relations (29.18), (29.19) it follows that equation (29.11) is equivalent with the relation

$$g(\underset{4}{k}) = Q\xi(\underset{4}{k}). \tag{29.24}$$

Observe that the functions $g(\underset{4}{k})$ occurring in (29.22) and (29.23) are identical.

Insertion of the expression (29.22) into (29.24) yields the equation

$$g(\underset{4}{k}) = QZa(\underset{4}{k}) + QZ^{-}g(\underset{4}{k}) \tag{29.25}$$

for the function $g(\underset{4}{k})$ which should be solved for an arbitrarily prescribed incoming amplitude $a(\underset{4}{k})$. Successive insertion of the right member of (29.25) into itself gives a formal series expansion for the solution $g(\underset{4}{k})$. This formal series indicates that the function $g(\underset{4}{k})$ may be assumed to be of the form

$$g(\underset{4}{k}) = WZa(\underset{4}{k}) \tag{29.26}$$

in which the operator W is to be so determined that

$$W = Q + QZ^{-}W. \tag{29.27}$$

In fact, if W satisfies this equation, the function $g(\underset{4}{k})$ defined by (29.26) satisfies equation (29.25) and the function $\xi(\underset{4}{k})$ defined by (29.22) satisfies condition (29.24).

We assume that there is a class (Q) of operators Q for which equation (29.27) possesses a solution W also of the class (Q); furthermore we assume that there is a class (ϕ) of functions $\phi(\underset{4}{k})$ on which the operators WZ, WZ^{+} may be applied and which produce functions of the same class. Apparently, the theory of the wave equation (29.1) has not yet been developed by establishing such classes of functions. It would seem probable that this could be done. In the following we shall assume that equation (29.27) and certain similar equations occurring below possess appropriate solutions.

Suppose that a solution $g(\underset{4}{k})$ of equation (29.27) has been found. Then we introduce the function

$$b(\underset{4}{k}) = a(\underset{4}{k}) - g(\underset{4}{k}) \tag{29.28}$$

and verify from relation (29.17) that for the function $\xi(\underset{4}{k})$ which satisfies condition (29.22), relation (29.23) holds with this function $b(\underset{4}{k})$. Using (29.26) we can express the function $b(\underset{4}{k})$ in the form

$$b(\underset{4}{k}) = (1 - WZ)a(\underset{4}{k}) \tag{29.29}$$

in terms of the function $a(\underset{4}{k})$. The operator $1 - WZ$ thus transforms the amplitudes of the incoming free field into the amplitudes of the outgoing free field. Properly speaking, the amplitudes of the incoming and outgoing free fields are given by the values of $a(\underset{4}{k})$ and $b(\underset{4}{k})$ for $\underset{4}{k}^2 = \mu^2$. Since only these values of $a(\underset{4}{k})$ enter the term $WZa(\underset{4}{k})$, the restricted values of $b(\underset{4}{k})$ depend only on the restricted values of $a(\underset{4}{k})$. The transformation of these restricted values of $a(\underset{4}{k})$ into those of $b(\underset{4}{k})$, given by the operator $1 - WZ$, *is, therefore, the scattering operator associated with the—unquantized—wave problem considered.* On occasion we shall use the term scattering operator for the unrestricted operator $1 - WZ$.

Inverse Operators

We proceed to establish important properties of the operator W. First we introduce the solution $W^{(1)}$ of the equation

$$W^{(1)} = Q + W^{(1)}Z^+Q, \tag{29.30}$$

which we assume to exist, just as we have assumed the solution W of equation (29.27) to exist. Multiplying equation (29.30) by $1 + W^{(1)}Z^+$ on the left we find

$$W^{(1)}(1 + Z^+W) = (1 + W^{(1)}Z^+)Q(1 + Z^+W) = (1 + W^{(1)}Z^+)W$$

or

$$W^{(1)} = W. \tag{29.31}$$

In other words, in addition to (29.27), the operator W satisfies the equation

$$W = Q + WZ^+Q. \tag{29.32}$$

At the same time it follows that the solution of equation (29.32) is unique; similarly, one deduces the uniqueness of the solution of (29.27).

Also we introduce the operator V as the solution of the equation

$$V = Q + QZ^{-}V \tag{29.33}$$

and establish in a similar way that it also satisfies the equation

$$V = Q + VZ^{-}Q. \tag{29.34}$$

We assume that solutions of these equations exist; we then know that they are unique.

Multiplying equation (29.32) by $1 + Z^{-}V$ on the right and equation (29.33) by $1 + WZ^{+}$ on the left we find

$$W(1 + Z^{-}V) = (1 + WZ^{+})Q(1 + Z^{-}V) = (1 + WZ^{+})V$$

or, by (29.17)

$$W - V + WZV = 0. \tag{29.35}$$

Similarly, we derive from (29.27), (29.34) and (29.17) the relation

$$W - V + VZW = 0. \tag{29.36}$$

Relations (29.35) and (29.36) lead to the relations

$$(1 - WZ)(1 + VZ) = 1, \tag{29.37}$$

$$(1 + VZ)(1 - WZ) = 1, \tag{29.38}$$

which express the fact that the operator $1 + VZ$ is the inverse of the scattering operator $1 - WZ$. Clearly, the operator $1 + VZ$ transforms the outgoing amplitudes $b(\underset{4}{k})$ into the incoming amplitudes $a(\underset{4}{k})$.

Inner Products

In order to formulate another important property of the operators W and V we introduce the inner product

$$\psi \cdot \phi = \int \psi(-\underset{4}{k})\phi(\underset{4}{k})\, d\underset{4}{k} \tag{29.39}$$

for functions ϕ and ψ of the class (ϕ). We do not require though, that the integral be defined for all functions in this class.

Let O be any operator acting on functions of $\underset{4}{k}$; then the operator $\tilde{O}$ will be called the transpose of O if the relation

$$O\psi \cdot \phi = \psi \cdot \tilde{O}\phi \tag{29.40}$$

holds whenever these inner products are defined. Evidently the kernel of $\tilde{O}$ is then the transpose of that of O,

$$(29.40)' \qquad (\underset{4}{k} \mid \tilde{O} \mid \underset{4}{k'}) = (\underset{4}{k'} \mid O \mid \underset{4}{k}).$$

From relations (29.7), (29.8) we infer that the operator Q is its own transpose,

$$(29.41) \qquad \tilde{Q} = Q.$$

From the definitions (29.12), (29.15) we infer that the transpose of the operator Z is $-Z$,

$$(29.42) \qquad \tilde{Z} = -Z.$$

From the definitions (29.14), (29.16) we infer that the operators $Z^{\pm}$ are transposed to each other,

$$(29.43) \qquad \tilde{Z}^{+} = Z^{-}, \qquad \tilde{Z}^{-} = Z^{+}.$$

From relations (29.43), (29.41) we see that the equations (29.33), (29.34) are the transposes of equations (29.32), (29.27). Since the solutions of these equations are unique it is clear that V is the transpose of W and vice versa,

$$(29.44) \qquad \tilde{W} = V, \qquad \tilde{V} = W.$$

It will be useful to make the scattering operator $1 - WZ$ the transpose of its inverse $1 + VZ$. This can be done by introducing the integral

$$(29.45) \qquad \psi \cdot Z\phi = 2\pi i \int \psi(-\underset{4}{k}) \operatorname{sgn} k_0 \delta(\underset{4}{k}^2 - \mu^2)\phi(\underset{4}{k})\, d\underset{4}{k}$$

as a new inner product. From relations (29.44), (29.42) we immediately deduce that $1 + VZ$ is the transpose of $1 - WZ$ with respect to this new inner product*

$$(29.46) \qquad (1 - WZ)\psi \cdot Z\phi = \psi \cdot Z(1 + VZ)\phi.$$

The value of the new inner product depends only on the restricted values of $a(\underset{4}{k})$ and $b(\underset{4}{k})$; using (29.12) we see that it can be written in the form

$$(29.47) \qquad \psi \cdot Z\phi = 2\pi i \int [\psi(-\omega, -\underset{3}{k})\phi(\omega, \underset{3}{k}) - \psi(\omega, \underset{3}{k})\phi(-\omega, -\underset{3}{k})]\, d\underset{3}{k}(2\omega)^{-1},$$

after substituting $-\underset{3}{k}$ for $\underset{3}{k}$ in the second contribution.

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*It could be shown that $1 + VZ$ is the Hermitian adjoint of $1 - WZ$ with respect to the Hermitian inner product

$$\int \overline{\psi}(\underset{4}{k})\zeta(\underset{4}{k})\phi(\underset{4}{k})\, d\underset{4}{k};$$

accordingly, these operators are unitary.

For application to quantized field problems it is convenient to assign to the function $\phi(\underset{4}{k})$ the two functions

$$(29.48) \qquad \phi^-(\underset{3}{k}) = \phi(\omega, \underset{3}{k}), \qquad \phi^+(\underset{3}{k}) = -\phi(-\omega, -\underset{3}{k})$$

(note that $\phi^+(\underset{3}{k}) = \overline{\phi^-(\underset{3}{k})}$ by (29.4) if the Fourier transform of $\phi(\underset{4}{k})$ is real) and similarly to assign $\psi^{\pm}$ to ψ. The inner product (29.47) then becomes

$$(29.49) \qquad \psi\cdot Z\phi = -2\pi i \int [\psi^+(\underset{3}{k})\phi^-(\underset{3}{k}) - \psi^-(\underset{3}{k})\phi^+(\underset{3}{k})]\, d\underset{3}{k}(2\omega)^{-1}.$$

Except for the factor $-2\pi i$, the right member here is exactly the inner product $\psi \circ \phi$ employed in Part V, Section 23, if $d\underset{3}{k}(2\omega)^{-1}$ is considered as the measure differential $dm(\underset{3}{k})$. Thus, as Zumino has observed, this inner product, introduced in a somewhat artificial way in Part V, occurs quite naturally in the present context.

30. *Boson Field Subject to Homogeneous Forces*

We are now ready to treat the problem of a boson field which corresponds to the problem of the unquantized field considered in Section 29. Operators

$$\Xi(\underset{4}{x}), \qquad \Xi(\underset{4}{k}), \qquad A(\underset{4}{k}), \qquad B(\underset{4}{k})$$

will take the place of the quantities $\xi(\underset{4}{x})$, $\xi(\underset{4}{k})$, $a(\underset{4}{k})$, $b(\underset{4}{k})$; they are supposed to satisfy the same linear equations as these unquantized quantities. The operators $A(\underset{4}{k})$ and $B(\underset{4}{k})$, however, will be defined only for values of $\underset{4}{k}$ on the hyperboloid $\underset{4}{k}^2 = \mu^2$.

From formula (29.20) we see that the operator $\Xi(\underset{4}{x})$ which corresponds to the incoming free field is expressed in terms of the operators $A(\underset{4}{k})$ as

$$(30.1) \qquad \Xi(\underset{4}{x}) = \theta^3 \int A(\underset{4}{k}) \exp\{-i\,\underset{4}{x}\,\underset{4}{k}\} \operatorname{sgn} k_0\, \delta(\underset{4}{k}^2 - \mu^2)\, d\underset{4}{k}.$$

The commutation law (3.13) for a free field can be seen to be equivalent with the commutation law

$$(30.2) \qquad [\zeta(\underset{4}{k'})A(\underset{4}{k'}),\ A(-\underset{4}{k''})\zeta(\underset{4}{k''})] = -2\pi i\ \delta(\underset{4}{k'} - \underset{4}{k''})\zeta(\underset{4}{k'})$$

or, with the law

$$(30.3) \qquad [\psi\cdot ZA,\ A\cdot Z\phi] = -2\pi i\ \psi\cdot Z\phi,$$

in which the inner product $\psi\cdot Z\phi$ introduced above proves useful.

The restricted values of the operators $A(\underset{4}{k})$, which alone enter these relations, will be identified with annihilation and creation operators by the formulas

(30.4) $$A^-(\underset{3}{k}) = A(\omega, \underset{3}{k}), \qquad A^+(\underset{3}{k}) = -A(-\omega, -\underset{3}{k})$$

which correspond to formulas (29.48). As seen from formula (29.21), expression (30.1) for the incoming free field becomes

(30.5) $$\Xi(\underset{4}{x}) = \theta^3 \int [A^-(\underset{3}{k}) \exp\{i\,\underset{3}{x}\,\underset{3}{k} - it\omega\} + A^+(\underset{3}{k}) \exp\{it\omega - i\,\underset{3}{x}\,\underset{3}{k}\}]\, d\underset{3}{k}(2\omega)^{-1}.$$

This expression agrees with expression (9.5) if $dm(\underset{3}{k}) = d\underset{3}{k}(2\omega)^{-1}$ is taken as measure differential. Note that $A^+(\underset{3}{k})$ is the adjoint of $A^-(\underset{3}{k})$ if $\Xi(\underset{4}{x})$ is self-adjoint and vice versa.

To verify the identification (30.4) one should note that the Fourier transformation associated with the measure differential $d\underset{3}{k}(2\omega)^{-1}$ must be written as

$$f(\underset{3}{x}) = \theta^3 \int g(\underset{4}{k}) \exp\{i\,\underset{3}{x}\,\underset{3}{k}\}\, d\underset{3}{k}(2\omega)^{-1/2}$$

in order that

$$\int |\, f(\underset{3}{x})\,|^2\, d\underset{3}{k} = \int |\, g(\underset{3}{k})\,|^2\, d\underset{3}{k}(2\omega)^{-1}.$$

Formula (30.5) then expresses the fact that $\Xi(\underset{4}{x})$ is the sum of the Fourier transforms of $(2\omega)^{-1/2} \exp\{-it\omega\}A^-(\underset{3}{k})$ and the complex conjugate Fourier transform of $(2\omega)^{-1/2} \exp\{it\omega\}A^+(\underset{3}{k})$, as required by formula (9.5).

The commutation law (30.2) for the restricted operators $A(\underset{4}{k})$ is equivalent with the commutation law

(30.6) $$[A^+(\underset{3}{k}'),\, A^-(\underset{3}{k}'')] = 2\omega'\,\delta(\underset{3}{k}' - \underset{3}{k}'').$$

This formula agrees with formula (8.16) since $2\omega'\delta(\underset{3}{k}' - \underset{3}{k}'')$ is the δ-function associated with the measure differential $d(\underset{3}{k})\ (2\omega)^{-1}$.

It is evident from the definition (30.4) that *the operators* $A^{\pm}(\underset{3}{k})$ *are invariant under Lorentz transformation when they are considered as functions of points* $\underset{4}{k}$ *on the half-hyperboloid* $\underset{4}{k}^2 = \mu^2$, $k_0 \geq 0$. The invariance of the commutation law (30.6) is consistent with the invariance of the differential $d\underset{3}{k}(2\omega)^{-1}$ under Lorentz transformation.

The operators $A^*(\underset{3}{k})$ belong to the incoming field. Operators $B^*(\underset{3}{k})$ associated with the outgoing field can be defined by formulas similar to (30.4) in terms of the operator $B(\underset{4}{k})$. The latter is given as $(1 - WZ)A(\underset{4}{k})$ in terms of the operator $A(\underset{4}{k})$. Using relation (29.46) and relation (30.3) for $(1 + VZ)\psi$, $(1 + VZ)\phi$ instead of ψ and ϕ one finds that the operators $B(\underset{4}{k})$ satisfy the same commutation law (30.3) as the operators $A(\underset{4}{k})$. In other words, *the operators* $B^*(\underset{3}{k})$ *satisfy the same commutation laws as the operators* $A^*(\underset{3}{k})$ *and therefore they may be interpreted as creation and annihilation operators for the outgoing free field.*

The argument just given is identical with the argument given in Part V, Section 23. The operator $1 - WZ$ may be interpreted as a "pseudo-operator" transforming the pair $A = \{A^*\}$ into the pair $B = \{B^*\}$. The fact that the operator $1 - WZ$ is the transpose of its inverse with respect to the inner product $\psi \cdot Z\phi$ shows that the pseudo-operator $\mathcal{Y}$ is the transpose ${}'\mathcal{Y}^{-1}$ of its inverse $\mathcal{Y}^{-1}$ with respect to the inner product $\psi \circ \phi$. The theory developed in Part V can therefore be applied.

The *canonical transformation* T which can be set up according to this theory transforms the incoming into the outgoing annihilation and creation operators, $B = T^{-1}AT$, and thus *serves as the field scattering operator* in the sense of Yang and Feldman (See Section 27 and [47]). The E-ordering of the transformation T with the aid of the Baker-Hausdorff theorem, cf. Section 24, enables one to give a detailed analysis of this transformation. In this way, for example, a basic formula for an electron field derived by Feynman with the aid of graphs [45] was derived by Zumino in a more direct way and written in a more concise form.

COMMENTS AND CORRECTIONS

As indicated in the preface, the actual content of the five parts differs somewhat from what was announced in the introduction. Below we enumerate the most important changes.

Parts I and II

p. 2: In Part III we shall show that a scattering operator is still possible if the external charges vary in time in an appropriate manner. Also we give a Lorentz invariant treatment of the problem.

The new type of field treated in Part IV and called myriotic there has as its most significant property that it possesses no vacuum state.

The treatment of electron-positron fields in Part V is actually as explicit as that of boson fields. An approximation procedure may be used for the unquantized problem but need not be used for the quantized problem.

Part VI is omitted.

A Lorentz invariant treatment of the problem of Part V, due to B. Zumino, is described in an Appendix.

p. 4, line 5 from the bottom: *read*

$$c_\sigma = (\Phi_\sigma\,,\ \Phi)(\Phi_\sigma\,,\ \Phi_\sigma)^{-1}.$$

p. 11, end of the introduction: Nevertheless, Dirac's notation will occasionally be used in later sections.

p. 14, last paragraph:

A formulation of commutation laws without use of the δ-function was already employed by Born and Infeld (See [53]).

Distributions in the sense of L. Schwartz were recently employed in an analysis of operators of quantum field theory by Güttinger [58].

p. 16: In fact, it is possible to ascribe a meaning to certain expressions of higher degree in $\Xi(x)$, $\Pi(x)$, which occur in problems involving nonlinear interaction not treated in this book.

p. 18: The functions $\Gamma_{\pm}$ are also denoted by $\pm\Delta^{\text{ret}}$, $\pm\Delta^{\text{adv}}$, respectively. The function $\tilde{\Gamma}$ is in general denoted by Δ if $\mu \neq 0$ and by D if $\mu = 0$.

p. 37, line 1: *replace* Section 16 by Section 19.

p. 60: An equivalent "front door" definition of the integral which is close to the definition by Kolmogorov will be given in a forthcoming paper by H. Shapiro and the author. The relationship to the Wiener integral will be explained there.

Part III

p. 69: In Part V we use the notation

$$b^+A^- \quad \text{for} \quad (b,\ A) \quad \text{and} \quad A^+b^- \quad \text{for} \quad (A,\ b).$$

p. 70, (13.22), third formula: *write*

$$V(t) = -J \cdots .$$

p. 71, (13.36), second formula: *write*

$$A_0^+(s) \quad \text{instead of} \quad A_0^+ ,$$

third formula: *substitute* q for j in two places.

p. 75: An "ultraviolet" catastrophe would occur if $(q, \Omega q) = \infty$ so that the asymptotic expectation values of the energies H and V become infinite, cf. (13.49), (13.50). For, if $(q, q) = 0$, the infinity of $(q, \Omega q)$ is due to the behavior of $q(k)$ for large values of k.

It was pointed out by van Hove [56] that in such a case the interaction energy V cannot even be defined for a single state $\Phi \neq 0$. In somewhat generalized form van Hove's theorem is:

Let $\gamma(s)$ be a function with $(\gamma, \gamma) = \infty$. Let $\gamma_n(s) = \gamma(s)$ in a subdomain $\mathcal{R}_n$ of the s-space and $\gamma_n(s) = 0$ outside $\mathcal{R}_n$. Assume $(\gamma_n, \gamma_n) < \infty$. Then for every state $\Phi \neq 0$

$$\{(A, \gamma_n) + (\gamma_n, A)\}\Phi \rightarrow \infty$$

as the subdomains $\mathcal{R}_n$ tend to cover the whole s-space.

This result expresses the fact that $\{(A, \gamma) + (\gamma, A)\}\Phi$ is not defined for $\Phi \neq 0$. With $\gamma = \Omega q$ we have $V = (A, \gamma) + (\gamma, A)$, cf. (13.22). The condition $(\Omega q, \Omega q) = \infty$ is even somewhat weaker than the condition $(q, \Omega q) = \infty$.

p. 86: The identity of the two forms of T could also be derived from the Baker-Hausdorff theorem employed in Part V, Section 23.

p. 91, (14.46)*, last term: *replace*

$$\psi_m^-(s)_r \quad \text{by} \quad \chi_m^-(s)_r .$$

p. 104, lines 15 and 17: *replace* (14.44) and (15.38) by (14.42) and (14.42)*, respectively.

p. 110, (16.30):

A few remarks may be made about the notion of "scattering operator".

First let the operator T_0^t with $T_0^0 = 1$ be defined as the solution of the Schrödinger equation

$$i\nabla_t T_{t_0}^t = H(t, t_0) T_{t_0}^t ,$$

in which $H(t, t_0)$ is the Hamiltonian assumed to be a function of the time t and expressed as a function of the operators $A^{\pm}(t_0)$. In the present problem we may set

$$H(t, t_0) = H(t_0) - \int A^+(s, t_0)\Omega^s q^-(s, t)\, dm(s) - \int q^+(s, t)\Omega^s A^-(s, t_0)\, dm(s),$$

where $H(t_0)$ is the undisturbed Hamiltonian

$$H_0(t_0) = \int A^+(s, t_0)\Omega^s A^-(s, t_0)\, dm(s).$$

We introduce the incomplete scattering operator

$$S_{t_0}^{t_1} = \exp\{i(t_1 - t_0)H_0(t_0)\}T_{t_0}^{t_1} ;$$

and note that it obeys the composition rule

$$S_{t_0}^{t_1}S_{t_1}^{t_2} = S_{t_0}^{t_2} .$$

Assuming that the operator $S_{t_0}^{t_1}$ has a limit as $t_0 \to \infty$ and $t_1 \to \infty$, we may define this limit S as the "scattering operator",

$$S_{t_0}^{t_1} \to S \qquad \text{as} \qquad t_0 \to \infty,\ t_1 \to \infty .$$

Assuming further that the adjusted operators

$$\exp\{\mp itH_0(0)\}A^{\pm}(t)$$

have limits $A_{\pm\infty}^{\pm}$ as $t \to \pm\infty$, the relation

$$A_{\infty}^{\pm} = S^{-1}A_{-\infty}^{\pm}S$$

holds. Clearly, this relation implies that the $A_{-\infty}$-representers of the state $S\Phi$ are the A_{∞}-representers of the state Φ. Therefore, S is the scattering operator in the sense used by Yang and Feldman, cf. Part V, Section 27 and [48].

In the problem of Section 16 the operators $A_{-\infty}^{\pm}$ are denoted by $A_0^{\pm}$ (see (16.4)). The operator S could be shown to differ from the operator $U(\infty)$, defined by (16.30), by a phase factor,

$$U(\infty) = S \exp\left\{\frac{i}{2}\gamma\right\},$$

where γ is given by (16.35) or the formula preceding (16.34).

In Part V, Section 27, a different operator is used as "scattering operator"; we denote it here by

$$\hat{S} = S_0^{\infty}S_{-\infty}^{0} .$$

This operator differs from S, but is related to $S = S_{-\infty}^{0}S_0^{\infty}$ by the transformation

$$\hat{S} = S_0^{\infty}SS_{-\infty}^{0} .$$

Since the $A_{-\infty}$-representers of the state $S_{-\infty}^{0}\Phi$ are the $A(0)$-representers of the state Φ, it follows that the matrix which furnishes the $A(0)$-representation of the operator $\hat{S}$ is the $A_{-\infty}$-representer of the operator S. This representer is nothing but the "S-matrix".

Incidentally, instead of scattering matrix, the term "collision matrix" is also used.

Part IV

p. 142: The situation described here is quite different from the situation described by van Hove in connection with an ultraviolet catastrophe. In that case, cf. comment to p. 75, the modified energy operator cannot be applied on any of the states on which the unmodified energy operator can be applied. In other words, two different Hilbert spaces of states are needed for the two different energy operators. Since a physical entity is characterized by the manifold of states in which it can be, two different physical entities are involved. Both Hilbert spaces can be considered as subspaces of a universal space introduced by von Neumann [54].

In the case treated here, however, the states on which the modified and the unmodified operators act belong to the same Hilbert space and hence refer to the same physical entity, even if the field is myriotic with respect to the modified and unmodified annihilation and creation operators and amyriotic with respect to the unmodified ones.

Part V

p. 185: The notion "homogeneous" refers to the fact that the associated differential equation is linear homogeneous and does not imply that the forces are constant in space.

p. 233: For the notion of scattering operator see the comments made in connection with Part III, (16.30), p. 46.

Appendix

p. 1: See comment to p. 185.

Supplementary Bibliography

[53] Born, M. and Infeld, L., *On the quantization of the new field theory*, Proceedings of the Royal Society, Volume 147, 1934, p. 522, and Volume 150, 1935, p. 141.

[54] von Neumann, J., *On infinite direct products*, Compositio Mathematica, Volume 6, Fasc. 1, 1938, pp. 1–77.

[55] Schwinger, J., *Quantum electrodynamics, I*, Physical Review, Volume 74, 1948, p. 1439; *II*, Volume 75, 1949, p. 651.

[56] van Hove, L., *Sur l'opérateur Hamiltonien de deux champs quantifiés en interaction*, Académie Royale de Belgique, Bulletin. Classe des Sciences, Volume 37, 1951, pp. 1055–1072; *Les difficultés de divergence pour un modelle particulier de champs quantifié*, Physica, Volume 18, 1952, pp. 145–159.

[57] Cook, J. M., *The mathematics of second quantization*, Transactions of the American Mathematical Society, Volume 74, No. 2, 1953, pp. 222–245.

[58] Güttinger, W., *Quantum field theory in the light of distribution analysis*, Physical Review, Volume 89, 1953, p. 1004.